Bifurcation analysis of an opinion dynamics model coupled with an environmental dynamics

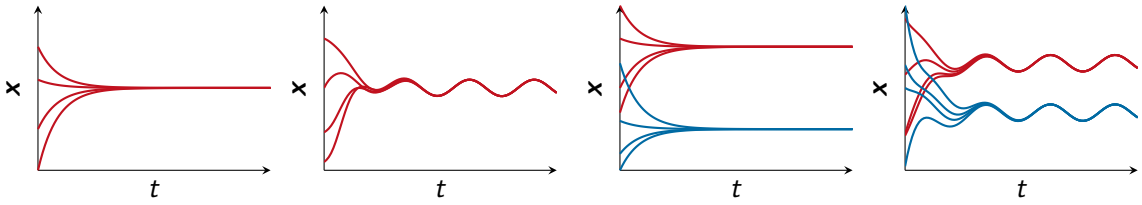
Anthony Couthures[1], Anastasia Bizyaeva[2], Vineeth S. Varma[1,3], Alessio Franci[4], Irinel-Constantin Morărescu[1,2]

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CDC

Understand the evolution of opinions

Opinion evolve in time: but how?



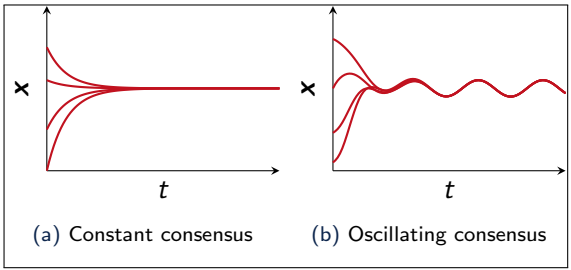
(a) Constant consensus (b) Oscillating consensus (c) Constant polarized (d) Oscillating polarized

Example of opinion evolution:

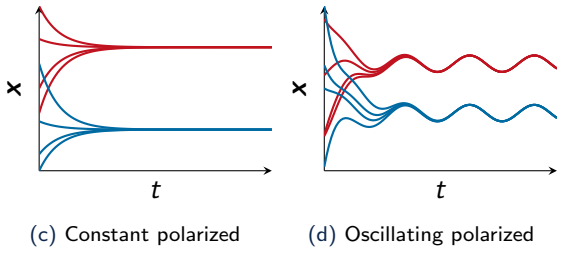
- Some movies are bad and everybody agree → Constant consensus
- During Covid-19 pandemic getting better or worse, → Oscillating (polarized?)

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(a) Constant consensus (b) Oscillating consensus



(c) Constant polarized (d) Oscillating polarized

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Outline

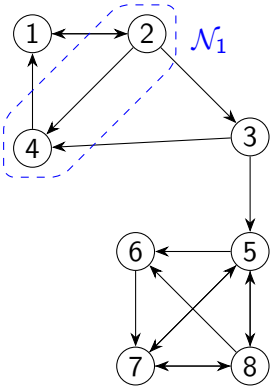
1. Model presentation
   Multi-agent with nonlinear communication
   Environmental dynamics and influence
   Coupling between Opinion and Environment

2. Bifurcation Analysis

3. Conclusion

Basic communication graph

- N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- $\mathcal{G}$ interaction graph fixed (directed or not).
- $\mathcal{N}_i$ neighborhood of i are agents influencing i.
- $\boldsymbol{A} = (a_{ij})_{i,j\in\mathcal{V}}$ adjacency matrix
  where $a_{ij} = 1$ if $i \leftrightarrow j$, 0 otherwise.
- $\boldsymbol{D} = \mathrm{diag}\,(d_i)_{i\in\mathcal{V}}$ degree matrix
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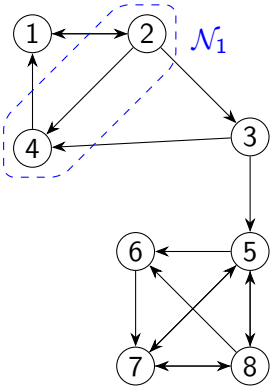
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In the context of Opinion dynamics

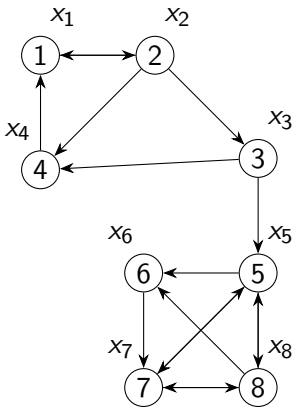
Agents are interacting:

- Social networks.
- Directed communication (media).



Communication graph with states

- Each node i has an state $x_i \in [-1, 1]$.
  In general, state is bounded in a compact set
  $\rightarrow$ we can normalize it to fit $[-1, 1]$.
- Vector notation $\boldsymbol{x} = (x_i)_{i \in \mathcal{V}}$.



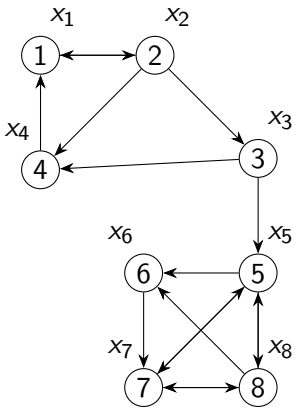
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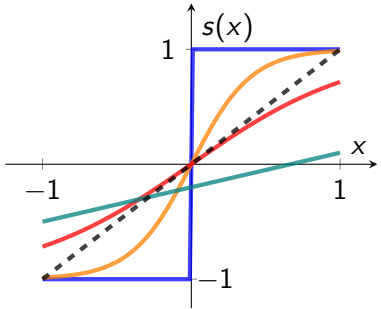
Agents have opinions on issues:

- Nonzero opinion: from "favorable" to "against".
- Opinions are personal and private.



Nonlinear communication with states

- Node communicates with signals/actions.
- i send signal $s(x_i)$ to its neighborhood $\mathcal{N}_i$.
- **Common** signal $s : [-1, 1] \to [-1, 1]$:
  smooth nonlinear, K-**Lipschitz**, **nondecreasing**.
- Vector notation $\boldsymbol{s}(\boldsymbol{x}) = (s(x_i))_{i \in \mathcal{V}}$.



Per node dynamics:

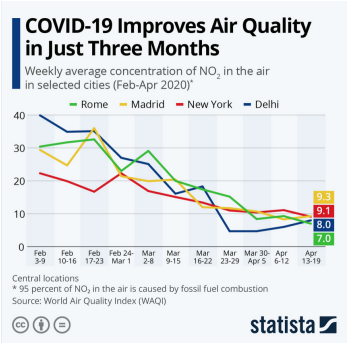
$$\dot{x}_i = \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} s(x_j) - x_i, \quad \text{each agent normalized by its degree.}$$

Similar to Opinion Dynamics [Fontan, 2017], [Franci, 2017], [Bizyaeva, 2023].
Can approximate quantization model [Martins, 2008], [Frasca, 2012], [Chowdhury, 2016],...

Environmental dynamics and influence

- Agents share a common environment e.
- They **influence collectively** e via $u(\boldsymbol{x}) \in [u_{\min}, u_{\max}]$.
- The environment evolves at time scale $\tau_e > 0$ according to

$$\tau_e \dot{e} = -\gamma e + u(\boldsymbol{x})$$

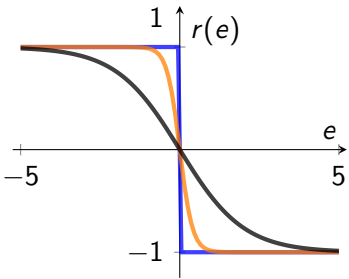


Simplified model inspired by [Berkowicz, 2000], [Garg, 2021], etc.

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- Feedback signal $r : \mathbb{R} \to [-1, 1]$ **decreasing**.
- Threshold at 0: $r(e) > 0$ if $e < 0$ and $r(e) < 0$ if $e > 0$.

Simplified model inspired by [Berkowicz, 2000], [Garg, 2021], etc.

Coupling the OD and Environment

The opinion of each agent evolves according to the dynamics:

$$\tau_x \dot{x}_i = -x_i + \underbrace{\beta r(e) + (1-\beta)}_{\text{Coupling part}} \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} s(x_j),$$

where

- $\tau_x > 0$ is the time constant for the opinion dynamics
- $\beta \in [0, 1]$ quantifies the trade-off between environmental and social influences.

Example: $\beta < 0.5$, agents follow **social** / $\beta > 0.5$ care about **the environment**

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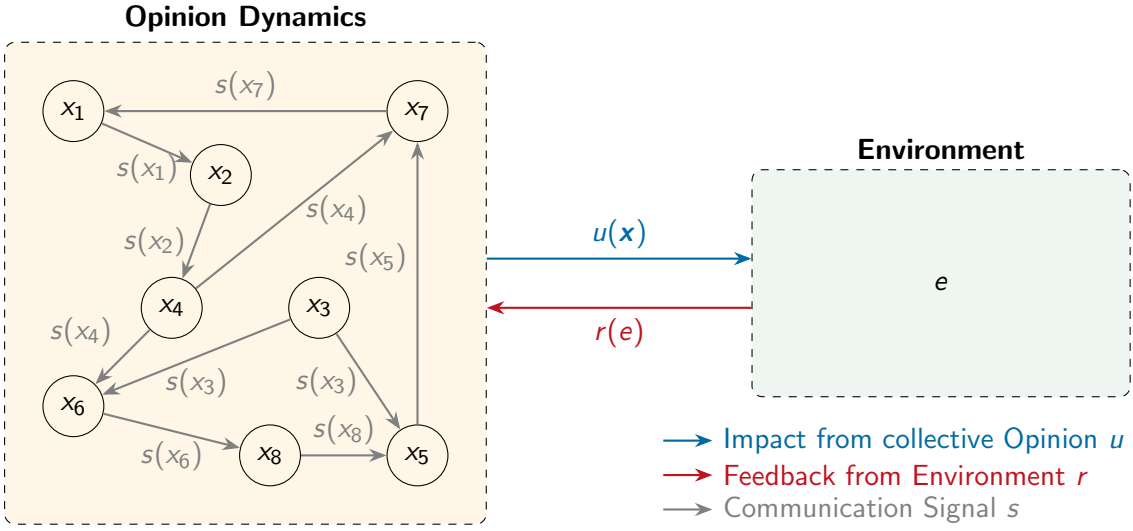
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The influence of the environment is uniform: same β and same response $r(e)$ for all agents.

Diagram of the coupled dynamics



Outline

1. Model presentation

2. Bifurcation Analysis
   - Model reduction
   - Numerical evidence
   - The analysis in a nutshell
   - Analytical Conditions & Consequences
   - Bifurcation Illustration

3. Conclusion

Model reduction: the FSO

We use the forward invariance of the Fully Synchronized Opinion (FSO) manifold

$$\mathcal{S} = \left\{ (\boldsymbol{x}, e) \in \mathcal{X}^N \times \mathbb{R} \mid \boldsymbol{x} = p\mathbb{1},\, p \in \mathcal{X} \right\}.$$

- $p \in [-1, 1]$ represents **the shared opinion.**

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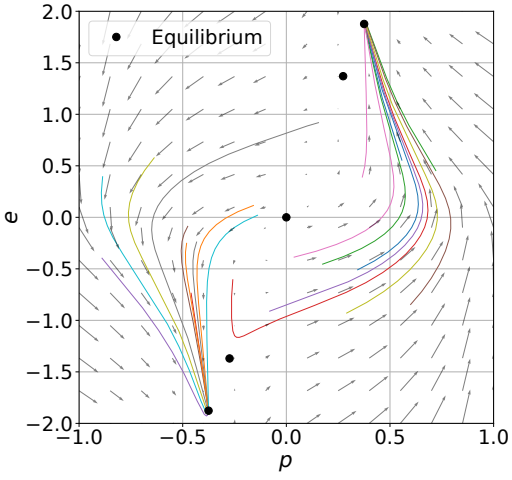
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Proposition: If $\mathcal{G}$ is a connected graph $\Rightarrow$ $\mathcal{S}$ is forward invariant.

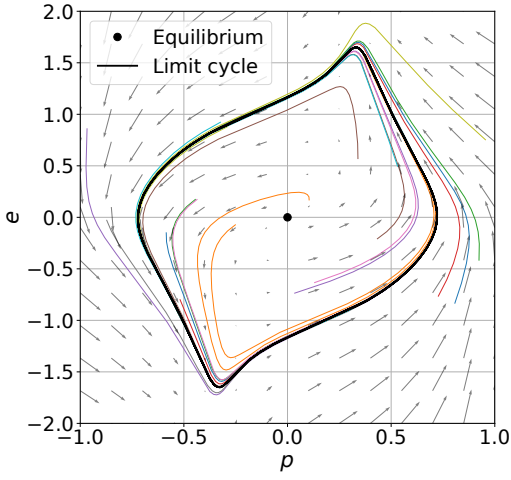
The dynamics reduces to

$$\begin{cases} \tau_x \dot{p} = & -p + \beta r(e) + (1-\beta) s(p) \\ \tau_e \dot{e} = & -\gamma e + u(p) - \gamma \bar{e}. \end{cases}$$

Simulation: From equilibria to Oscillations

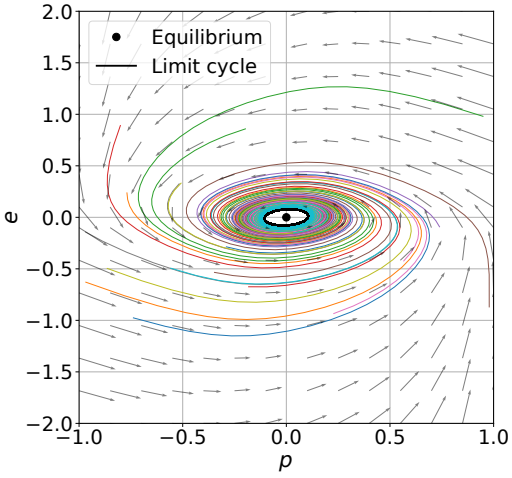


(a) $\beta = 0.24$

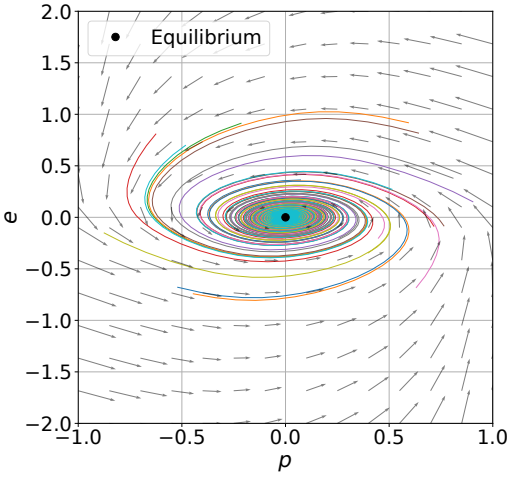


(b) $\beta = 0.25$

Simulation: From Oscillations to Equilibrium



(a) $\beta = 0.59$



(b) $\beta = 0.61$

The analysis in a nutshell

Under oddness assumption over s, r, u, the steps are:

1. **Characterization of Singular Points** of $D_y F(y, \beta, \gamma) \to$ Potential bifurcations points. The Jacobian matrix of F is given by:

$$D_y F(y, \beta, \gamma) = \begin{bmatrix} \tau_x^{-1} & 0 \\ 0 & \tau_e^{-1} \end{bmatrix} \begin{bmatrix} (1-\beta)s'(p) - 1 & \beta r'(e) \\ u'(p) & -\gamma \end{bmatrix}.$$

   Identifying singular points: where the Jacobian loses full rank.

The analysis in a nutshell

Under oddness assumption over s, r, u, the steps are:

1. **Characterization of Singular Points** of $D_y F(y, \beta, \gamma) \rightarrow$ Potential bifurcations points.
2. **Defining Pitchfork Conditions at** $0 \rightarrow$ The condition for a zero eigenvalue at $(0, 0)$. The Jacobian $D_y F(y, \beta, \gamma)$ has at least one zero eigenvalue iff

$$\gamma = -\frac{u'(p)r'(e)\beta}{(1-\beta)s'(p) - 1} \quad \text{and} \quad (1-\beta)s'(p) > \max(1, -u'(p)r'(e)\beta + 1).$$

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3. **Proving Pitchfork using Lyapunov-Schmidt Reduction** $\rightarrow$ Theorem 1

Theorem (Pitchfork bifurcation)

Under condition before at $y^* = (0, 0)$ *has a zero eigenvalue associated with the critical eigenvector* $\boldsymbol{v} = \begin{bmatrix} 1 & u'(0)/\gamma^* \end{bmatrix}^\top$.
Moreover, if

$$c = (1 - \beta^*)s'''(0) + \beta^* r'''(0)\frac{u'(0)^3}{{\gamma^*}^3} + \frac{\beta^* r'(0)}{\gamma^*}u'''(0)$$

is nonzero, then a pitchfork bifurcation occurs at y^* *as* β *passes through* β^*. *In particular, the bifurcating branches emerge along the subspace generated by* $\boldsymbol{v}$.

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4. **Defining Hopf Conditions** $\rightarrow$ The condition for two complex conjugate eigenvalues to have a zero real part....
5. **Hopf Validation (Transversality)** $\rightarrow$ Theorem 2 The existence of periodic orbits is proven by...

Analytical Conditions & Consequences

- **Necessity of Opinion Amplification ($s'(p) > 1$)**
  - $s'(p) > 1$ is the necessary condition for the Jacobian to become singular

$\rightarrow$ The signal function s must be an **amplifier** of opinions!! without this, no bifurcations (neither Pitchfork nor Hopf) can occur.

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- **Pitchfork Bifurcation: Symmetry Breaking and Bistability**
  - β variation trigger **sudden, symmetry-breaking shift**.

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- **Hopf Bifurcation: Recurrent Cycles**
  - β variation trigger **oscillation and limit cycle**.

$\rightarrow$ it leads to a predator-prey dynamics, rather than convergence to a static equilibrium.

Simulation: Bifurcation Diagram

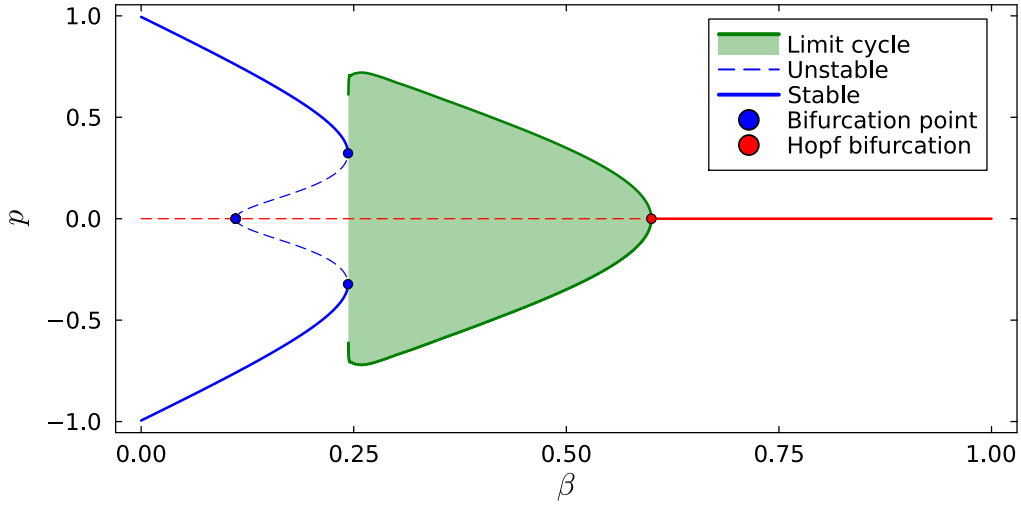


Figure: Bifurcation diagram according to the parameter β.

Outline

1. Model presentation
2. Bifurcation Analysis
3. Conclusion

Conclusion & Perspectives

Summary of Contributions

- **Coupled Modeling:** interplay between social influence and environmental feedback.
- **Role of Trust (β):** the key driver of qualitative changes, requiring opinion amplification to trigger bifurcations.
  - **Bistability (Pitchfork):** Explains sudden convergence to Pro- or Anti-environmental states.
  - **Oscillations (Hopf):** Models the recurrent cycles of environmental collapse and recovery.

Future Directions

- **Outside of $\mathcal{S}$ Analysis:** Extending the analysis to the full model: bifurcation in opinion.
- **Multiple Environment Bifurcations:** add other environments like **economy**, **health**,...
- **Heterogeneous OD:** examine how heterogeneities in β, $r(e)$, $S(x_i)$ between agents impact the feedback dynamics