Analysis of a continuous opinion and discrete action dynamics model coupled with an external observation dynamics

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Motivation

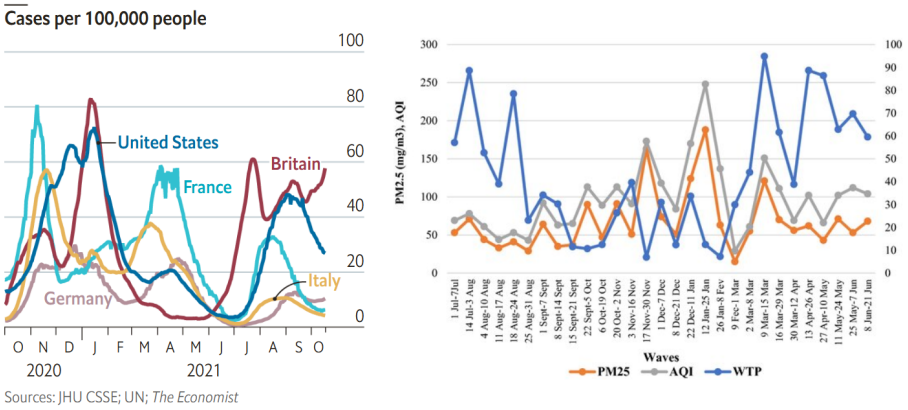


Figure: Wave like patterns for Covid-19 and pollution [J.He, 2021] in Environmental and Resource Economics

Key terms: WTP - Willingness to pay, PM2.5/AQI (pollution measures)

Outline

1. Introduction
   Opinion dynamics
   Model dynamics
2. Analysis of the coupled system
3. Conclusion and perspective

Basic Interaction graph

Agents are interacting:

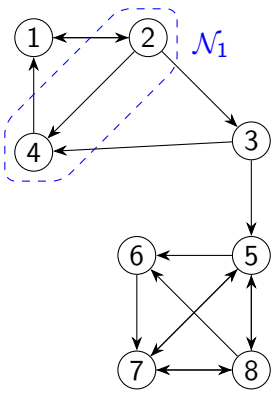
- Social networks.
- Directed communications (medias).

Basic Interaction graph

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- Directed communications (medias).

- N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- Interaction graph fixed $\mathcal{G}$ (directed or not).
- Neighborhood of i is agents influencing i noted $\mathcal{N}_i$.
- Cardinal of $\mathcal{N}_i$: $|\mathcal{N}_i| = n_i$.



Interaction graph with Opinion

Agents have opinions on issues:

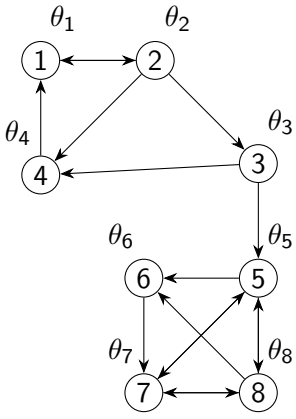
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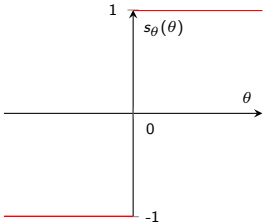
- Each node i has an opinion $\theta_i \in [-1, 1]$.
- Vector notation $\boldsymbol{\theta} = (\theta_i)_{i \in \mathcal{V}}$.
- Adjacency matrix $\boldsymbol{A}$ and degree matrix $\boldsymbol{D}$.



Discrete Signal Dynamics

- Node communicates with signals/actions.
- Node i sends signal $s_\theta(\theta_i)$ to neighborhood $\mathcal{N}_i$.
- Vector notation $\boldsymbol{s_\theta}(\boldsymbol{\theta}) = (s_\theta(\theta_i))_{i\in\mathcal{V}}$.

$$s_\theta(\theta(k)) = \begin{cases} 1 & \text{if } \theta_i(k) > 0 \\ -1 & \text{if } \theta_i(k) < 0 \end{cases}$$



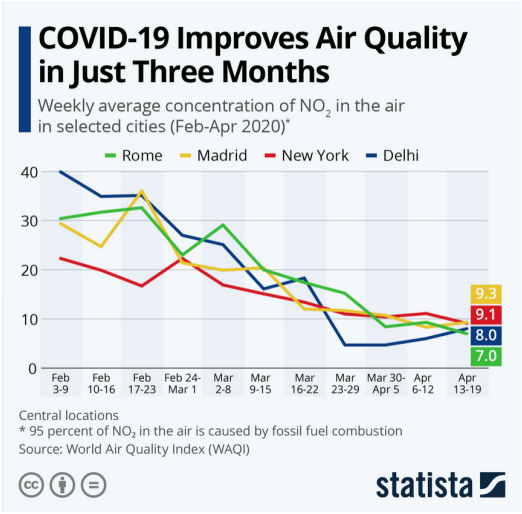
Graph dynamics:

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) + \left(\mathbf{I} - \operatorname{diag}\left(\boldsymbol{\theta}(k)\right)^2\right)\left(\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{s}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}(k)\right) - \boldsymbol{\theta}(k)\right)$$

Inspired by CODA model of [Martins, 2008], OD with tunable sensitivity [Anastasia, 2002] etc.

Basic concept

- Opinions impact their decisions, which affect the pollution.
- Social media/discussions and signals from the pollution also impact opinions.



Impact of Agents on Environment

- Agents impact a shared environment p.
- Agent i impact is quantified by $e(s_\theta(\theta_i))$: $e(-1) = e_{\text{min}}$, $e(1) = e_{\text{max}}$.
- We note $e_i(k)$ emission of agent i at time k.
- The environment's state evolves according to:

$$p(k+1) = (1-\gamma)p(k) + \sum_{i=1}^{N} e_i(k).$$

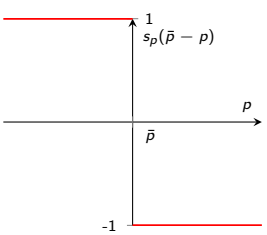
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$$p(k+1) = (1-\gamma)p(k) + \sum_{i=1}^{N} e_i(k).$$

- Feedback signal $s_p : \mathbb{R} \to [-1, 1]$ based on threshold $\bar{p}$:

$$s_p(p(k)) = \begin{cases} -1 & \text{if } p(k) > \bar{p} \\ 1 & \text{if } p(k) < \bar{p} \end{cases}$$



Here p is the climate state and e are emissions. Simple model inspired by [Berkowicz, 2000], [Garg 2021] etc.

Coupled Dynamics

- We denote by s_θ and s_p, the signals from agents and the environment.
- β the weight between neighbors opinion and external signal.

Then the dynamics become:

- Graph:

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) + \left(\mathbf{I} - \operatorname{diag}\left(\boldsymbol{\theta}(k)\right)^2\right) \left[\beta s_p(\bar{p} - p(k))\mathbb{1} + (1-\beta)\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{s}_\theta\left(\boldsymbol{\theta}(k)\right) - \boldsymbol{\theta}(k)\right]$$

With environment:

$$p(k+1) = (1-\gamma)p(k) + \sum_{i=1}^{N} e_i(k).$$

Outline

1. Introduction

2. Analysis of the coupled system
   - Solution Space
   - Preservation of action
   - Clusters in signal/action
   - Fully Synchronized Opinion (FSO)

3. Conclusion and perspective

Dynamics reformulation in discrete action case

Let $n^+(k)$ and $n^-(k)$ be the number of positive and negative opinion at k. Then,

$$p(k+1) = (1-\gamma)p(k) + n^+(k)e_{\max} + n^-(k)e_{\min}.$$

Let

$$\lim_{k\to\infty} n^+(k) = n^+, \quad \lim_{k\to\infty} n^-(k) = n^-.$$

In this case the equilibrium p^* is given by

$$p^* = \frac{n^+ e_{\max} + n^- e_{\min}}{\gamma}.$$

with associated signal s_p^*.

Set containing Equilibrium

Proposition

Let $i \in \mathcal{V}$ and assume that nobody start as extremist. If $(s_p(\bar{p} - p(k)))_{k\geq 0}$ and $\left(n_i^+(k)\right)_{k\geq 0}, \left(n_i^-(k)\right)_{k\geq 0}$ are stationary sequences with limit s_p^* and n_i^{+*}, n_i^{-*}, respectively. Then opinion $(\theta_i(k))_{k\geq 0}$ converges to

$$\theta_i^* = \lim_{k\to\infty} \theta_i(k) = (1-\beta)\,\frac{n_i^{+*} - n_i^{-*}}{n_i} + \beta s_p^* \in \mathbb{Q} + \beta\mathbb{Q}$$

$\Rightarrow$ Equilibrium depends on trade-off between pros/cons. and climate.

We search for oscillation and for not steady states.

Preservation of actions/signals

Idea: Looking at equilibrium in signal/action space instead of state space.

i.e. $s_\theta(\theta_i(k)) = s_\theta(\theta_i(k+1))$.

If actions are not stable $\Rightarrow$ opinions are not stable.

Lemma

Let $i \in \mathcal{V}$ and assume that $\beta < 1/(1+n_i)$. The following statements hold:
- *if $n_i^+(k) > n_i^-(k)$ and $s_\theta(\theta_i(k)) = 1$ then, $s_\theta(\theta_i(k+1)) = 1$,*
- *if $n_i^-(k) < n_i^+(k)$ and $s_\theta(\theta_i(k)) = -1$ then, $s_\theta(\theta_i(k+1)) = -1$.*

$\Rightarrow$ For some range of parameters β, nodes follow the most present action.

Clusters pattern in signal/action

In CODA without environment there are cluster formation. Here as well!

Definition

$A \subset \mathcal{V}$ is a weakly (strongly) robust polarized cluster if:

- $\forall i, j \in A$, $s_\theta(\theta_i(0)) = s_\theta(\theta_j(0))$,
- $\forall i \in A$, $|\mathcal{N}_i \cap A| \geq |\mathcal{N}_i \setminus A| - (+)\beta/(1-\beta)|\mathcal{N}_i|$.

Lemma

A a robust polarized cluster then $\forall i \in A$, $\forall k \in \mathbb{N}$, $s_\theta(\theta_i(k)) = s_\theta(\theta_i(0))$.

$\Rightarrow$ Signal of agents in A are constant if climate never change signal (even if climate change signal).

Cluster Formation

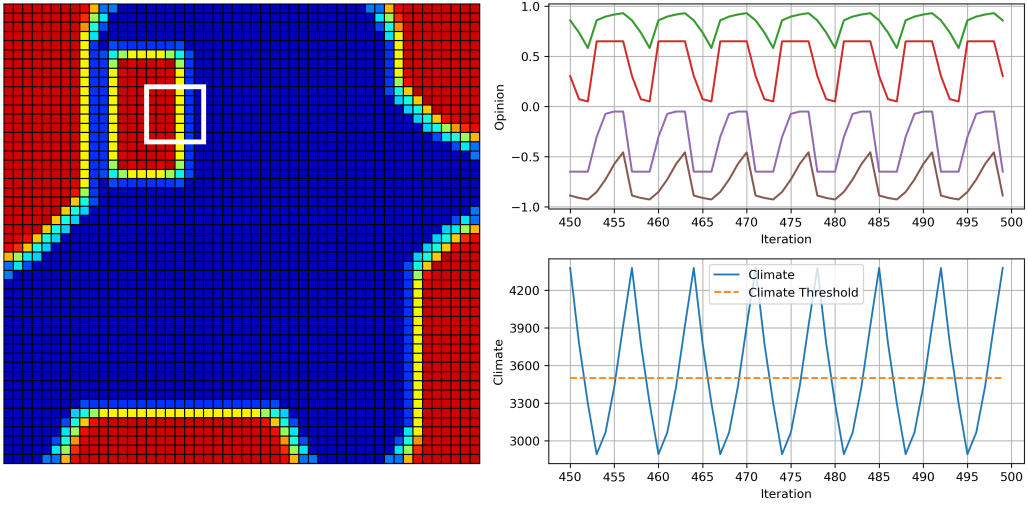


Figure: Robust polarized cluster on a grid lattice.

Fully Synchronized Opinion (FSO)

Definition

$\boldsymbol{\theta} = (\theta_1 \cdots \theta_N)^\top$ is Fully Synchronized (FS) if $\forall i, j \in \mathcal{V}$, $\theta_i = \theta_j$.

Proposition

Assume $\beta > 1/2$ and $\boldsymbol{\theta}$ being FS. The following statements hold true.

- $\{(\mathbb{1}, -1), (-\mathbb{1}, 1)\}$ *are not equilibrium in the action space.*
- $(\mathbb{1}, 1)$ *is stable in the action space* $\Leftrightarrow Ne_{max}/\gamma \leq \bar{p}$.
  $\Rightarrow (\boldsymbol{\theta}, p) = (\mathbb{1}, Ne_{max}/\gamma)$ *is a stable equilibrium.*
- $(-\mathbb{1}, -1)$ *is stable in the action space* $\Leftrightarrow Ne_{min}/\gamma \geq \bar{p}$.
  $\Rightarrow (\boldsymbol{\theta}, p) = (-\mathbb{1}, Ne_{min}/\gamma)$ *is a stable equilibrium.*

For some parameter no actions stable $\Rightarrow$ limit cycle or chaos!!!

Phase diagram for the 3 regimes in FSO

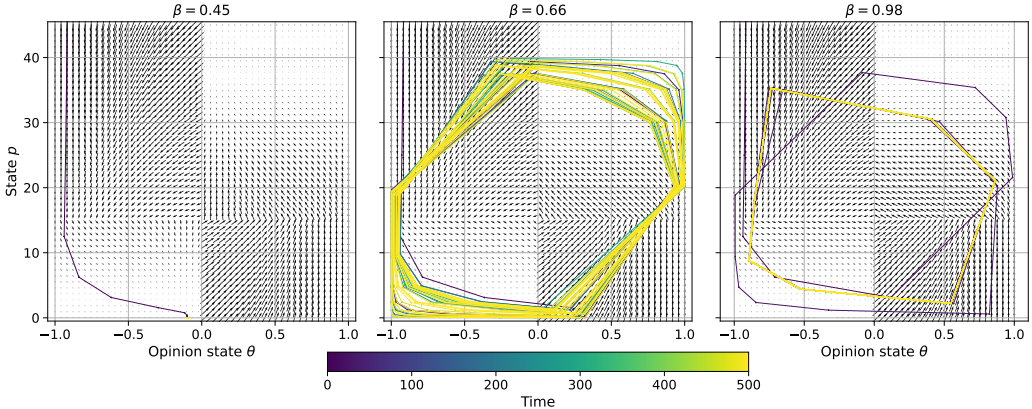


Figure: Behavior of trajectories for different value of β.

Bifurcation diagram for FSO

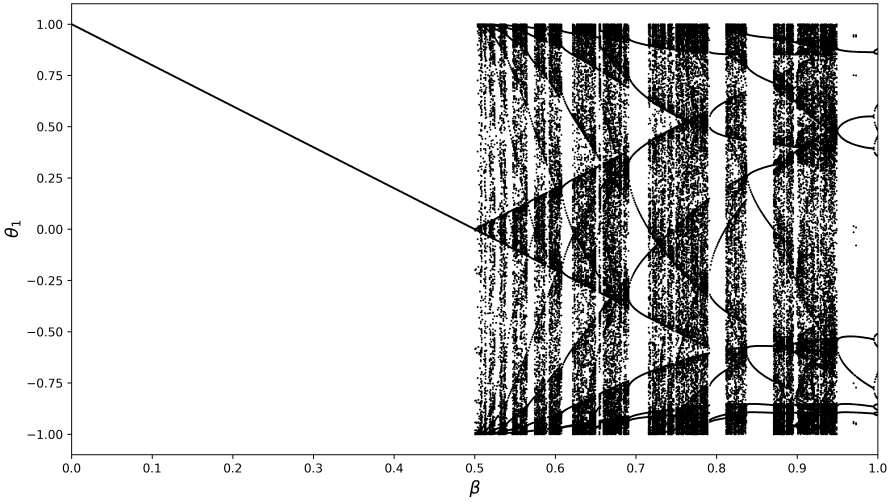


Figure: Bifurcation diagram with chaotic behaviors of opinion in FSO for $\beta > 1/2$.

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Conclusion and perspective

Conclusion:

1. We studied a coupled dynamical system involving opinions (interacting over a graph) and an external signal.
2. We observe stable equilibria, limit cycles or chaos based on the strength of coupling.
3. Analysis of what seems to be a Prey-Predator system (Climate v.s. Opinion)

Perspectives:

- Considering other kind of signals (current work)
- Results characterizing chaos and limit cycles, control of the process etc.
- Look in the continuous time settings if there is chaos.