The link between equitable partitions and local agreements in multi-agent systems with nonlinear interactions

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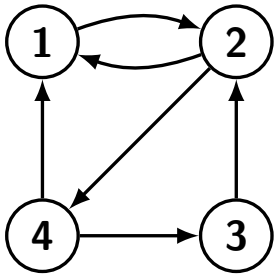






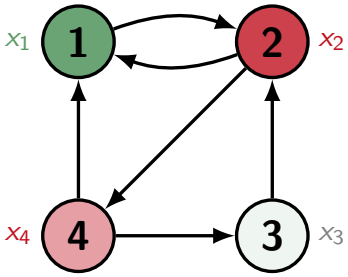
Basic Definition and Notation of Opinion Dynamics

- **Network:** N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- **Classic graph definition:**
  - **Adjacency** matrix $\boldsymbol{A}$ and **Degree** matrix $\boldsymbol{D}$
  - **Normalized adjacency** matrix $\boldsymbol{D}^{-1}\boldsymbol{A}$



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- **Opinion State:** Each node i holds a value x_i.
- **Normalized Opinion Space** $x_i \in [-1, 1]$:
  - $+\mathbf{1}$: Strongly Favorable (Green).
  - $-\mathbf{1}$: Strongly Against (Red).
- **State Vector** $\boldsymbol{x} = (x_i)_{i \in \mathcal{V}}$.



State Vector $\boldsymbol{x}$

$$\boldsymbol{x} = \begin{pmatrix} 0.9 \\ -0.8 \\ 0.1 \\ -0.4 \end{pmatrix} \begin{matrix} \leftarrow \text{Node 1} \\ \leftarrow \text{Node 2} \\ \leftarrow \text{Node 3} \\ \leftarrow \text{Node 4} \end{matrix}$$

Opinion Dynamics with Nonlinear Communication

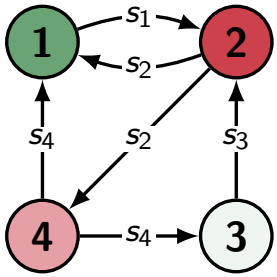
- Agents observe **signals** not opinions since opinions are private.
- **Signal** function $s : [-1, 1] \to [-1, 1]$.

Dynamics Per-agent:

$$\dot{x}_i = \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} \underbrace{s(x_j)}_{\text{Signal}} - x_i$$

Collective dynamics:

$$\dot{\boldsymbol{x}} = \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{s}(\boldsymbol{x}) - \boldsymbol{x}$$



References:

- (Couthures et al. LCSS–2025),
- (Couthures et al. arXiv–2025),

For signed graphs:

- (Fontan and Altafini 2018),
- (Bizyaeva et al. 2023),

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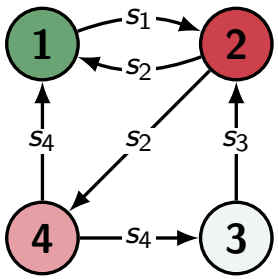
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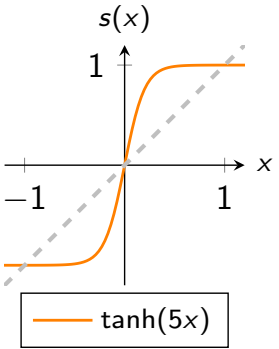
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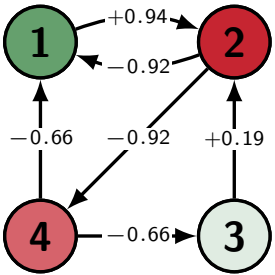


- The signal s is **common** and K-**Lipschitz**, **nondecreasing**.
- **Fixed** graph **symmetric** and **positive**.

Opinion Dynamics with Nonlinear Communication

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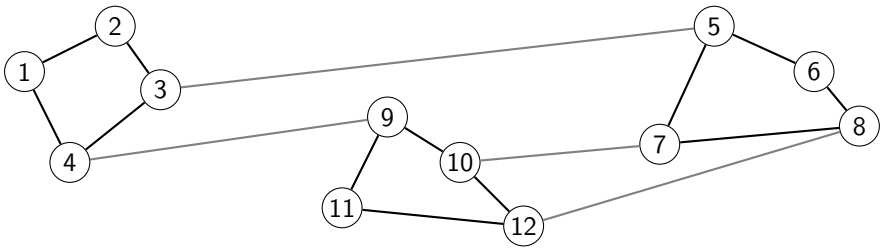


Example of signal:
Smooth sigmoidal influence.

With nonlinear signals:
Polarization is possible!!!

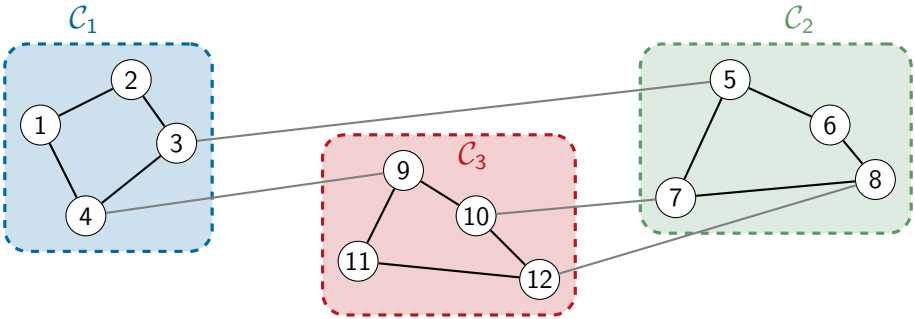
Graph partition

Take a graph $\mathcal{V}$,



Graph partition

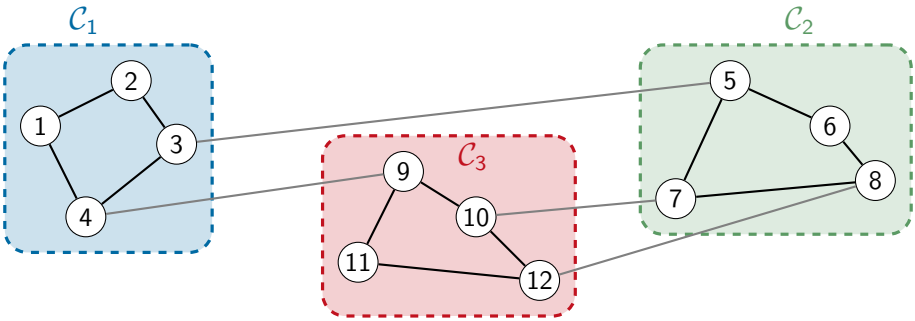
Take a graph $\mathcal{V}$, we cut the graph into "cells" to make a partition $\pi = \{\mathcal{C}_i\}_{i \in \mathcal{I}}$.



Question: When do agents reach perfect consensus in their own cell?

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Question: When do agents reach perfect consensus in their own cell?

Cell Synchronized Equilibria

$\boldsymbol{x}^*$ is *Cell Synchronized Equilibrium (CSE)* for partition π if all agents within each cell $\mathcal{C}_k$ have the same state value: $x_i = x_j, \forall i, j \in \mathcal{C}_k$.

Structural Implications of Cell Synchronization

At the equilibrium ($\dot{x}_i = 0$), the state of agent i depends on its neighbors $\mathcal{N}_i$:

$$x_i^* = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} s(x_j^*)$$

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Grouping by cells: Let $b_{il} = |\mathcal{N}_i \cap \mathcal{C}_l|$ be the number of neighbors of agent i in cell $\mathcal{C}_l$. Under the CS assumption ($x_i^* = y_k^*$), the equation becomes:

$$y_k^* = \sum_{l=1}^{M} \frac{b_{il}}{d_i} s(y_l^*)$$

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Necessary Condition: Weight Regular Partition

$$\frac{b_{il}}{d_i} = \frac{b_{i'l}}{d_{i'}}, \quad \forall l \in \{1, \ldots, M\} \text{ and } \forall i, i' \in \mathcal{C}_k$$

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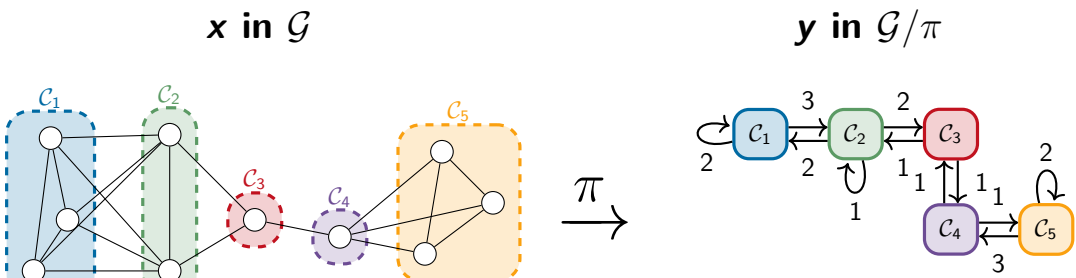
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Stronger Structural Condition: Equitable Partition

Require the number of connections between cells to be constant:

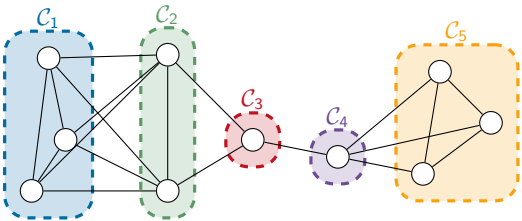
π is an **Equitable Partition (EP)** if $b_{il} = b_{i'l}, \quad \forall i, i' \in \mathcal{C}_k, \forall l \in \{1, \ldots, M\}$

Equitable Partition & Quotient Graph



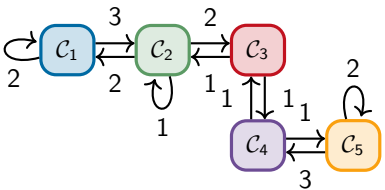
Equitable Partition & Quotient Graph

x in $\mathcal{G}$



π

y in $\mathcal{G}/\pi$



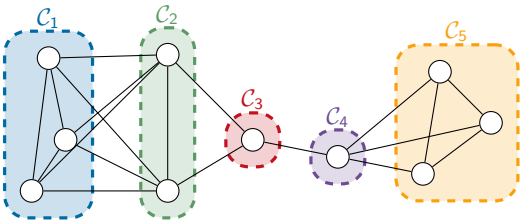
Characteristic Matrix P

Maps agents to their cells ($x^* = Py^*$)

$$P^\top = \begin{bmatrix} \mathbb{1}_{\mathcal{C}_1} & \mathbb{1}_{\mathcal{C}_2} & \mathbb{1}_{\mathcal{C}_3} & \mathbb{1}_{\mathcal{C}_4} & \mathbb{1}_{\mathcal{C}_5} \end{bmatrix}$$

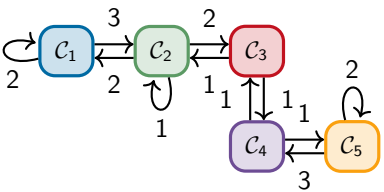
Equitable Partition & Quotient Graph

$\boldsymbol{x}$ **in** $\mathcal{G}$



π

$\boldsymbol{y}$ **in** $\mathcal{G}/\pi$



Characteristic Matrix $\boldsymbol{P}$

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Quotient Matrix $\boldsymbol{B}$

Number of edges from $\mathcal{C}_k$ to $\mathcal{C}_l$

$$\boldsymbol{B} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Cell-Synchronized Manifold

Definition: Cell Synchronized Manifold

Given a graph $\mathcal{G}$ with an EP π, the *Cell Synchronized Manifold* $\mathcal{S}_\pi$ is the subspace of states that are perfectly cell-synchronized:

$$\mathcal{S}_\pi = \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} \text{ for some } \boldsymbol{y} \in \mathcal{Y}\}$$

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Proposition: Forward Invariance

Let π be an Equitable Partition of the graph $\mathcal{G}$. The cell-synchronized manifold $\mathcal{S}_\pi$ is **forward invariant** with respect to the network dynamics.

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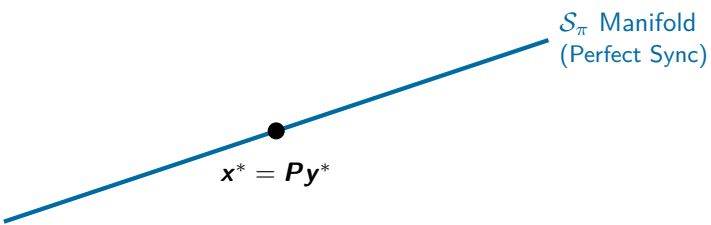
Let π be an Equitable Partition of the graph $\mathcal{G}$. The cell-synchronized manifold $\mathcal{S}_\pi$ is **forward invariant** with respect to the network dynamics.

How could we use that to prove stability of a CSE?

Decomposing the Stability Problem

To prove the local exponential stability of a CSE $\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{y}^*$, we do not need to analyze the massive $N \times N$ full system Jacobian.

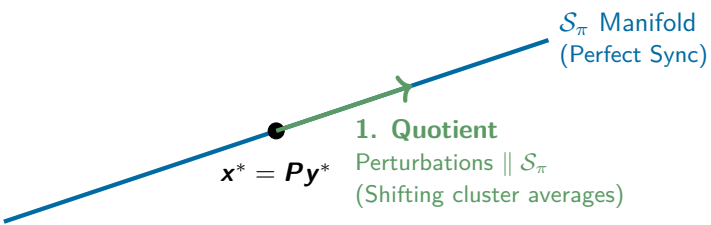
Instead, we leverage the forward invariance of $\mathcal{S}_\pi$ and decompose the stability into **two orthogonal ingredients**:



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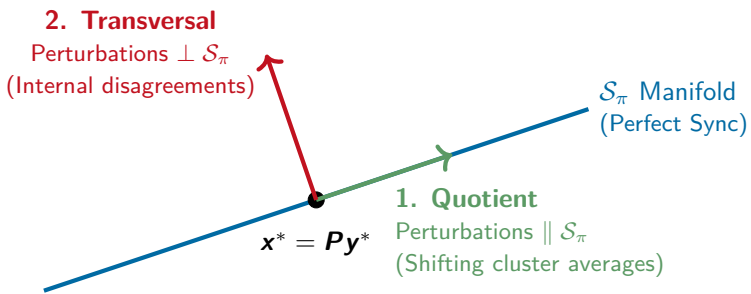
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Instead, we leverage the forward invariance of $\mathcal{S}_\pi$ and decompose the stability into **two orthogonal ingredients**:



Quotient Stability (Along $\mathcal{S}_\pi$)

One can notice the following:

CSE and the quotient dynamics

$\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{y}^*$ is a CSE in $\mathcal{G}$ if and only if its cell values $\boldsymbol{y}^*$ is an equilibrium in the quotient graph $\mathcal{G}/\pi$.

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If we perturb the system while keeping cells perfectly synchronized, the stability relies entirely on the **Quotient Jacobian**:

$$\boldsymbol{J}_{\text{quot}}(\boldsymbol{y}^*) = \underbrace{\tilde{\boldsymbol{D}}^{-1}\boldsymbol{B}}_{\text{Quotient Graph}} \overbrace{\boldsymbol{s}'(\boldsymbol{y}^*)}^{\text{Signal Slope}} - \boldsymbol{I}_M$$

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Quotient stability condition

For $\boldsymbol{y}^*$ to be locally stable we need: $\boldsymbol{J}_{\text{quot}}(\boldsymbol{y}^*)$ must be Hurwitz.

Intuition: *The connections between cells* $\boldsymbol{B}$ *and the signal slope* s' *must pull the cell averages to* $\boldsymbol{y}^*$ *locally.*

Transversal Stability (Attractivity of $\mathcal{S}_\pi$)

If agents desynchronize ($\perp \mathcal{S}_\pi$), the system must actively force them back into consensus.

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This requires two sub-conditions:

1. Internal Cohesion

Each cell $\mathcal{C}_k$ with interal adjacency $\boldsymbol{A}_k$ must pull its agents together. We define the *internal decay rate*:

$$\alpha_k(y_k^*) = 1 - \frac{s'(y_k^*)}{d_{\mathcal{C}_k}} \lambda_{N_k-1}(\boldsymbol{A}_k) > 0$$

Intuition: *The algebraic connectivity of the cell (λ_{N_k-1}) must overcome destabilization from the slope of the signal (s').*

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2. Bounded Inter-Cell Coupling

Disagreements shouldn't cascade. We define a coupling matrix $\boldsymbol{G} \in \mathbb{R}^{M\times M}$:

$$G_{kk} = -\alpha_k(y_k^*), \quad G_{kj} \propto \frac{s'(y_j^*)}{d_{\mathcal{C}_k}} \sqrt{b_{kj} b_{jk}}$$

Intuition: ***G*** *is small-gain w.r.t to internal disagreements to ensure the network globally dissipates energy.*

Main Theorem: Synthesis

By combining these two orthogonal properties, we obtain a lower-dimensional condition for the full N-dimensional network.

Theorem: Full Local Stability

Let $\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{y}^*$ be a Cell-Synchronized Equilibrium resulting from a connected Equitable Partition π.

If we have:

1. **Quotient stability:** $\boldsymbol{J}_{\text{quot}}(\boldsymbol{y}^*)$ is Hurwitz, AND
2. **Transversal stability:** Internal decay rate $\alpha_k(y_k^*) > 0$ for all cells, and the inter-cell coupling gain matrix $\boldsymbol{G}$ is Hurwitz.

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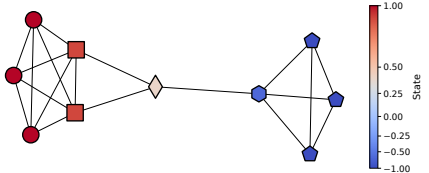
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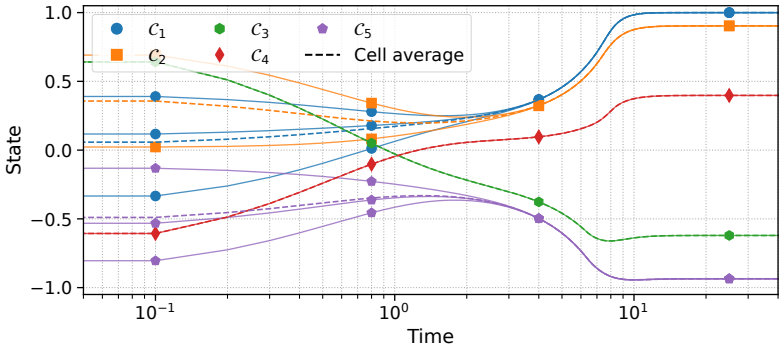
The stability analysis of the $N \times N$ system is reduced to an $M \times M$ quotient matrix and the local algebraic connectivity of individual cells!

Numerical Example

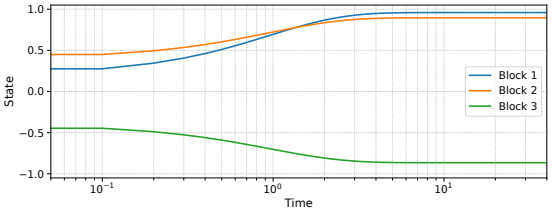
Graph:



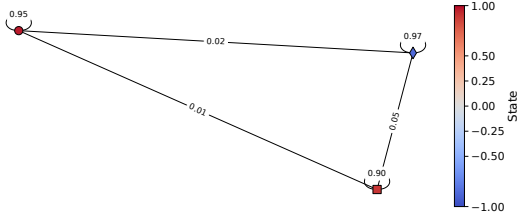
Trajectories:



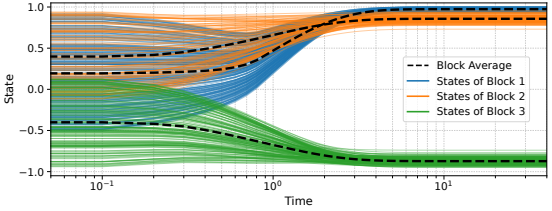
Numerical Perspective: Robustness for Approximate EPs



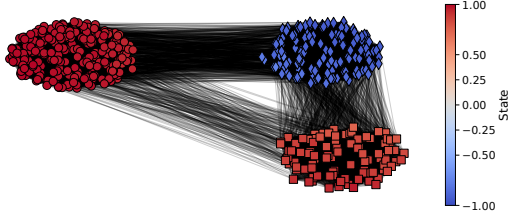
(a) **Quotient graph:** State trajectories over time.



(b) **Quotient graph topology**



(c) **Stochastic Block Model:** State trajectories over time.



(d) **Stochastic Block Model topology**

- Bizyaeva, Anastasia, Alessio Franci, and Naomi Ehrich Leonard (Mar. 2023). "Nonlinear Opinion Dynamics With Tunable Sensitivity". In: *IEEE Trans. Automat. Contr.* 68.3, pp. 1415–1430. ISSN: 0018-9286, 1558-2523, 2334-3303. DOI: 10.1109/TAC.2022.3159527.
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