Global synchronization of multi-agent systems with nonlinear interactions

A. Couthures, V.S Varma, S. Lasaulce, I.-C. Morărescu

CRAN, CNRS, Université de Lorraine, Nancy, France

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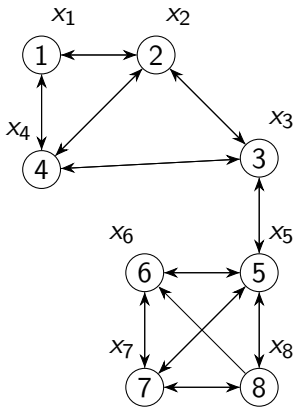




Multi-agent model

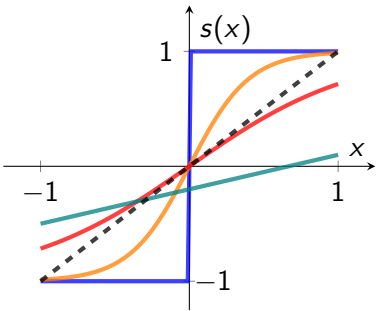
- N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- $\mathcal{G}$ interaction graph fixed.
- $\mathcal{N}_i$ neighborhood of i are agents influencing i.
- $\boldsymbol{A} = (a_{ij})_{i,j \in \mathcal{V}}$ adjacency matrix
  where $a_{ij} = 1$ if $i \leftrightarrow j$, 0 otherwise.
- $\boldsymbol{D} = \operatorname{diag}(d_i)_{i \in \mathcal{V}}$ degree matrix
  $d_i = \sum_{j=1}^{N} a_{ij} = |\mathcal{N}_i|$ is the degree of node i.

- Each node i has a state $x_i \in [-1, 1]$.
  In general, state is bounded in a compact
  $\rightarrow$ we can normalize it to fit $[-1, 1]$.
- Vector notation $\boldsymbol{x} = (x_i)_{i \in \mathcal{V}}$.



Nonlinear interactions

- Node communicates with signals/actions.
- i send signal $s(x_i)$ to its neighborhood $\mathcal{N}_i$.
- **Common** signal $s : [-1, 1] \to [-1, 1]$:
  (non)linear, K-**Lipschitz**, **nondecreasing**.
- Vector notation $\boldsymbol{s}(\boldsymbol{x}) = (s(x_i))_{i \in \mathcal{V}}$.



Node dynamics:

$$\dot{x}_i = \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} s(x_j) - x_i,$$

Similar to Opinion Dynamics [Fontan, 2017], [Franci, 2018], [Bizyaeva, 2023].
Can approximate quantized interactions [Martins, 2008], [Frasca, 2012], [Chowdhury, 2016],...

Collective dynamics

Framework Assumptions:

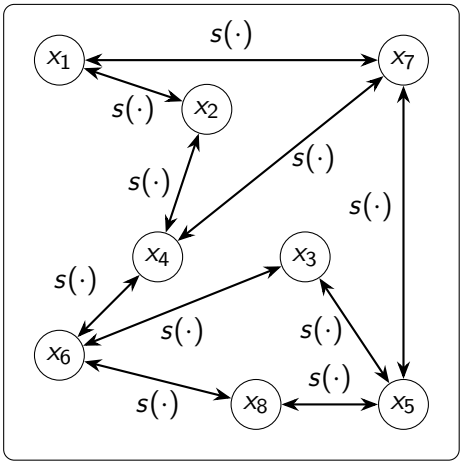
- $\mathcal{G}$ is a **undirected connected** graph.
- **Common** signal $s : [-1, 1] \to [-1, 1]$: K-**Lipschitz**, **nondecreasing**.

Collective dynamics:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{s}\left(\boldsymbol{x}\right) - \boldsymbol{x}. \\ &= -\boldsymbol{D}^{-1}\boldsymbol{L}\boldsymbol{s}\left(\boldsymbol{x}\right) + \boldsymbol{s}\left(\boldsymbol{x}\right) - \boldsymbol{x} \\ &= -\boldsymbol{D}^{-1}\boldsymbol{L}\boldsymbol{x} + \boldsymbol{D}^{-1}\boldsymbol{A}(\boldsymbol{s}\left(\boldsymbol{x}\right) - \boldsymbol{x}). \end{aligned} \tag{1}$$

All those writing are equivalent.

Nonlinear Consensus Dynamics



Invariance of the synchronization manifold

Definition (Synchronization / Synchronization manifold)

A state $\boldsymbol{x} \in [-1,1]^N$ is a *synchronization* if $x_i = x_j$ for all $i,j \in \mathcal{V}$. The *synchronization manifold* is

$$\mathcal{S} = \left\{ \boldsymbol{x} \in [-1,1]^N \mid \forall i,j \in \mathcal{V},\ x_i = x_j \right\}.$$

$\mathcal{S}_c$ is the synchronization at c. i.e., $\mathcal{S}_c = c\mathbb{1}$.

One have:

Lemma (Forward invariance)

The synchronization manifold $\mathcal{S}$ is forward invariant for the dynamics.

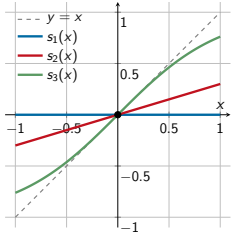
i.e., once the agents are synchronized, they will remain synchronized for all times.

Different types of signal functions

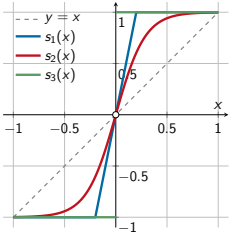
Definition (Estimations)

A signal function s is said to be:

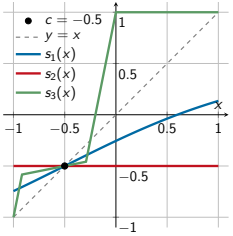
- an *global underestimation* if $x(s(x) - x) \leq 0$ for all $x \in [-1, 1]$.
- an *global overestimation* if $x(s(x) - x) \geq 0$ for all $x \in [-1, 1]$.
- a *consistent estimation* around $c \in \mathrm{Fix}(s)$ if there exists a neighborhood $I \subseteq [-1, 1]$ of c s.t for all $x \in I$, $(x - c)(s(x) - x) \leq 0$.



(a) Underestimations



(b) Overestimations



(c) Consistent estimations: $c = -0.5$.

Different types of signal functions

Definition (Estimations)

A signal function s is said to be:

- an *global underestimation* if $x(s(x) - x) \leq 0$ for all $x \in [-1, 1]$.
- an *global overestimation* if $x(s(x) - x) \geq 0$ for all $x \in [-1, 1]$.
- a *consistent estimation* around $c \in \mathrm{Fix}(s)$ if there exists a neighborhood $I \subseteq [-1, 1]$ of c s.t for all $x \in I$, $(x - c)(s(x) - x) \leq 0$.

Proposition (Global underestimation $\implies$ convergence to consensus)

Let $\mathcal{G}$ be a connected graph and s be a **globally underestimating** *signal function. Then, every solution of* (1) *approaches a synchronization equilibrium at a fixed point of s.*

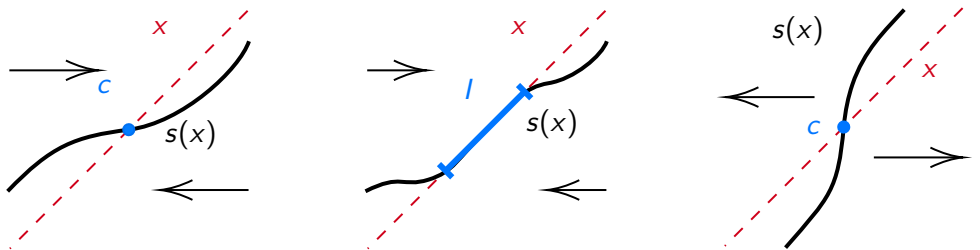
Agents will converge to a consensus equilibrium **at a fixed point of s**.

Synchronization induced by the signal function

Theorem (Stability of the consensus equilibrium)

For $\mathcal{G}$ connected graph and a synchronization equilibrium $\mathcal{S}_c = c\mathbb{1}$, the following holds true:

1. *$\mathcal{S}_c$ is locally stable if and only if s is a consistent estimation around c.*
2. *$\mathcal{S}_c$ is locally asymptotically stable if and only if $\mathcal{S}_c$ is locally stable and c is an isolated fixed point of s.*



Synchronization induced by the signal function: example

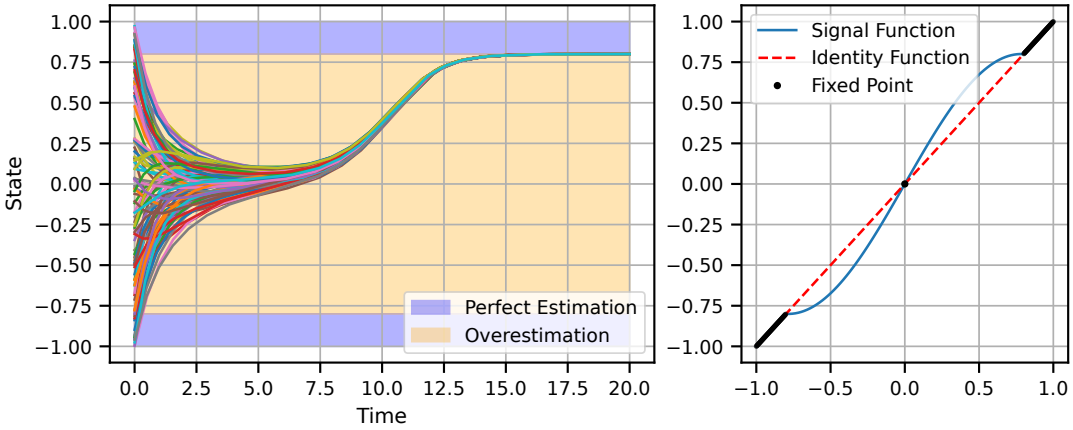


Figure: State evolution with a globally overestimating signal for random graph and i.i.d. initial condition.

Synchronization induced by the graph structure

Synchronization can also arise from **symmetries**:

Proposition (Same neighbors $\implies$ synchronization)

Let $i, j \in \mathcal{V}$ having the exact same neighbors (twin-nodes). Then, agents i and j asymptotically synchronize.

Corollary (All-to-all graph or complete bipartite $\implies$ synchronization)

Let $\mathcal{G}$ be the all-to-all graph or a complete bipartite graph. Then, all agents asymptotically synchronize.

Those results **do not depend on the signal function s!!**

Agents in a connected graph don't always agree

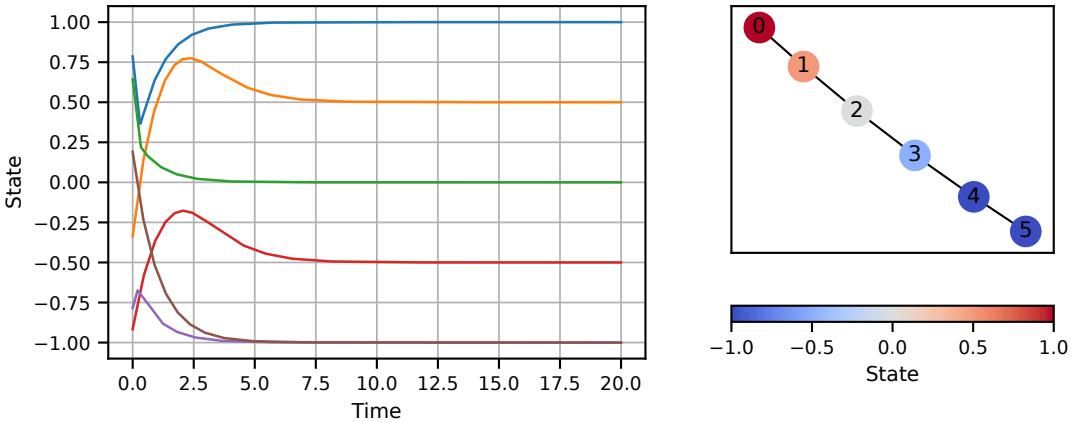


Figure: Random i.i.d. initial conditions for a line graph of 6 agents with **a globally overestimating signal** ($s(x) = \tanh(20x)$). The agents do not converge to a synchronization equilibrium.

Conclusion and perspective

Conclusion:

- We presented a **nonlinear consensus dynamics** that encapsulate the linear case, approximate quantization case, and can be used to model opinion dynamics.
- We showed that the dynamics can be globally/locally asymptotically stable to consensus, and that the convergence depends on the signal function and the graph structure.

Perspectives:

- Extend the results to directed graphs, signed graphs, and time-varying graphs.
- Study the local synchronization in the graph: trade-off between graph connectivity and signal function.
- Study the effect of the signal function on the convergence and community detection.