Opinion Dynamics with Nonlinear Interaction: From Robust Clustering to Environmental Coupling

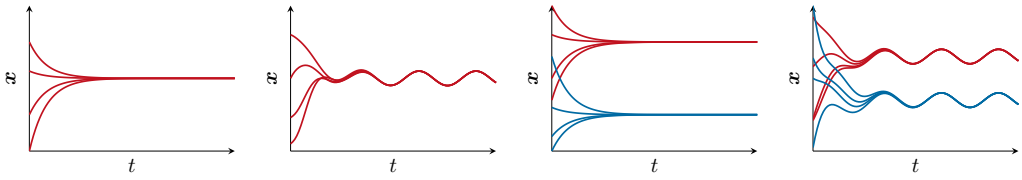
Anthony Couthures[1]

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18th January 2026 - Louvain-la-Neuve, Belgium

Understand the evolution of opinions

What do we observe?



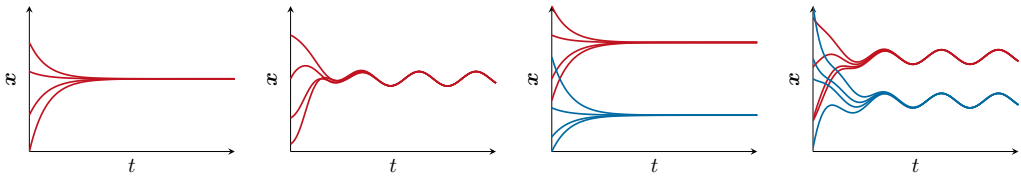
(a) Constant consensus (b) Oscillating consensus (c) Constant polarized (d) Oscillating polarized

Example of opinion evolution:

- Some movies are bad and everybody agrees → Constant consensus
- During Covid-19 pandemic getting better or worse → Oscillating (polarized?)

Understand the evolution of opinions

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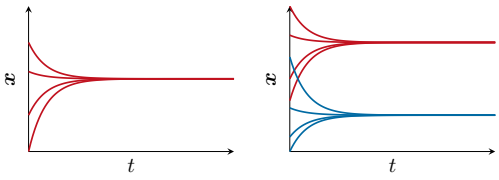
(a) Constant consensus (b) Oscillating consensus (c) Constant polarized (d) Oscillating polarized

Questions:

1. How do we shift from consensus to polarization?
2. How do we shift from constant consensus to oscillating consensus?

Understand the evolution of opinions

What do we observe?



(a) Constant consensus (b) Constant polarized

Questions:

1. How do we shift from consensus to polarization?

Outline

1. Opinion Dynamics presentation
   1.1 From Social Networks to Matrices
   1.2 Routes to Polarization
   1.3 Multi-agent with nonlinear communication

2. Reaching a Consensus

3. Emergence of Polarization

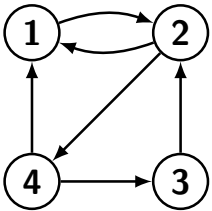
4. Coupling Opinion and Environment

5. Emergence of Oscillations

6. Conclusion

From Social Networks to Matrices

- **Network:** N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- $\mathcal{G}$ interaction graph fixed (directed or not).



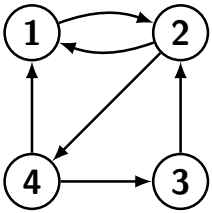
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In the context of Opinion dynamics

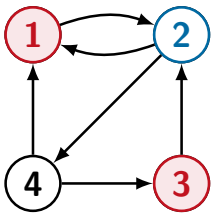
Agents are interacting:

- Social networks.
- Directed communication (media).



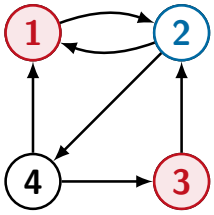
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  - Agents that **influence** i ($j \to i$).
  - **Example:** Focus on **Agent 2**.
    2 receives info from 1 and 3.
    $\mathcal{N}_2 = \{\mathbf{1}, \mathbf{3}\}$.



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- **Adjacency Matrix** $\boldsymbol{A}$ (Sensing):
  - $a_{ij} = 1$ if $j \to i$ (Col j affects Row i).
  - **Row 2** captures who 2 listens to.



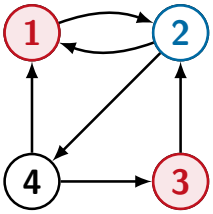
Adjacency $\boldsymbol{A}$
(Row i = Who i listens to)

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Row 2: 1s at indices 1 and 3

From Social Networks to Matrices

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- **In-Degree Matrix** $\boldsymbol{D}$:
  - $d_i = \sum_j a_{ij}$ (Row Sum).
  - $d_2 = 1 + 1 = 2$.



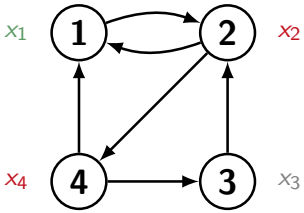
Degree $\boldsymbol{D}$
(In-Degree)

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$d_2 = |\mathcal{N}_2| = 2$

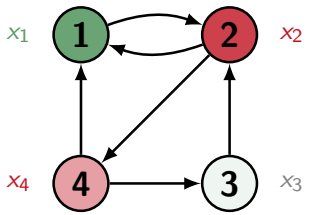
Associating States to Agents

- **State:** Each node i holds a value x_i.



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  - Normalized bounded set.
  - **+1:** Strongly Favorable (Green).
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  - **0:** Neutral (White).



Opinion Dynamics Context

- Opinions are **private** internal states.
- Agents update x_i based on neighbors.

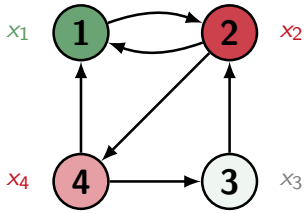
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- **State Vector** $\boldsymbol{x} = (x_i)_{i \in \mathcal{V}}$:
  - Stacks all opinions into one column.

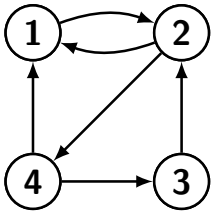


State Vector $\boldsymbol{x}$

$$\boldsymbol{x} = \begin{pmatrix} 0.9 \\ -0.8 \\ 0.1 \\ -0.4 \end{pmatrix} \begin{matrix} \leftarrow \text{Node 1} \\ \leftarrow \text{Node 2} \\ \leftarrow \text{Node 3} \\ \leftarrow \text{Node 4} \end{matrix}$$

Routes to Polarization

- **The Linear Trap:**
  - Standard consensus ($\dot{x} = -Lx$).
  - Converges to average value (Consensus).
  - *Cannot* model persistent disagreement.



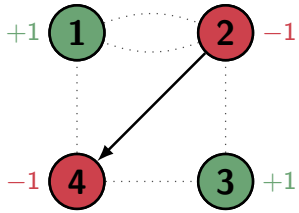
Linear Model

Connectivity leads to Consensus

Routes to Polarization

- **How to sustain Polarization?**
  Three main theoretical approaches:

1. **Dynamic Topology** (Bounded Confidence)
   - Cut edge if $|x_i - x_j| > R$.
   - Result: **Disconnected graph**.
   - Refs: [Hegselmann-Krause, 2002].



Bounded Confidence

Connections Cut

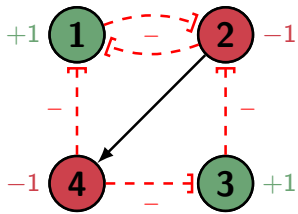
(Graph becomes disconnected)

Routes to Polarization

- **How to sustain Polarization?**
  Three main theoretical approaches:

1. **Dynamic Topology** (Bounded Confidence)
   (Edges cut based on opinion distance)

2. **Antagonistic Interactions**
   - Negative weights ($a_{ij} < 0$).
   - "Enemies" repel each other.
   - Refs: [Altafini, 2013].



Antagonistic
Negative Weights
(Structural Balance)

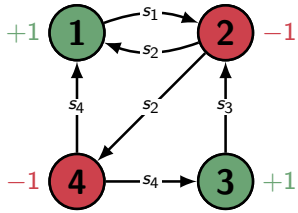
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   (Negative edge weights)
3. **Nonlinear Communication**

Our Focus

- **Fixed, connected graph**.
- **Positive weights** $a_{ij} > 0$.
- Polarization via signal $s(x)$.



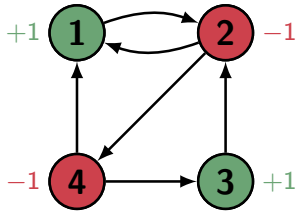
Nonlinear Model

Standard Connected Graph
(Polarization via Signal)

Routes to Polarization

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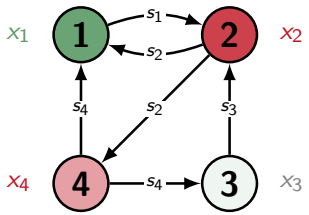
Main advantage:
If the signal are non-decreasing, then the system is cooperative and no attracting limit cycle exists.

Nonlinear Communication

- **Mechanism:** Agents communicate **signals**.
- i sends $s(x_i)$ to neighborhood $\mathcal{N}_i$.

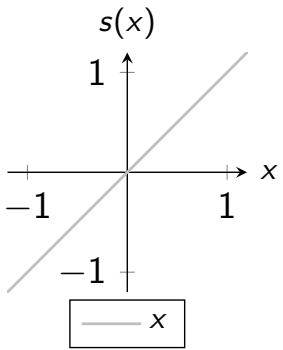
Opinion Dynamics Context

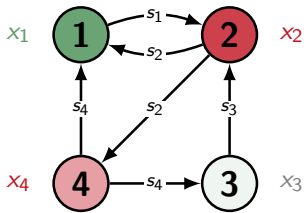
- Agents observe **Action**, not **Opinion**.
- Agents update x_i based on **Action** of their neighbors.



Nonlinear Communication

- **Mechanism:** Agents communicate **signals**.
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- **Signal** $s : [-1, 1] \to [-1, 1]$



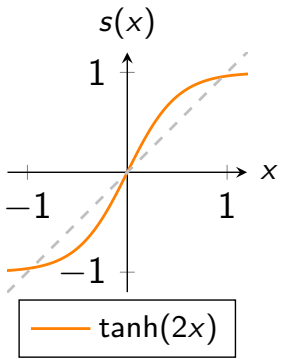


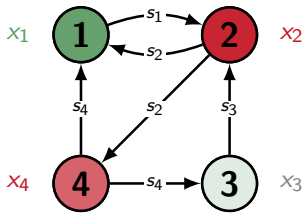
Key Feature
This is linear averaging.

References: [DeGroot, 1974]

Nonlinear Communication

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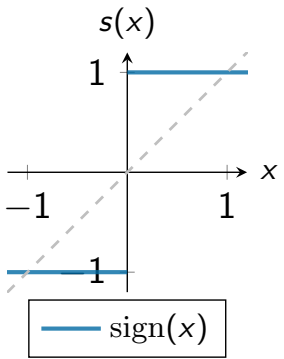


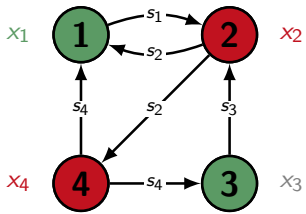
Key Feature
Influence is smooth sigmoidal.

References: [Fontan, 2017], [Franci, 2017], [Bizyaeva, 2023]

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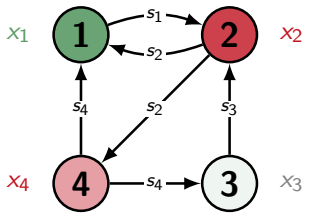


Key Feature
Signal is quantized: discrete values.

References: [Martins, 2008], [Frasca, 2012], [Chowdhury, 2016]

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Dynamics: Per-agent dynamics

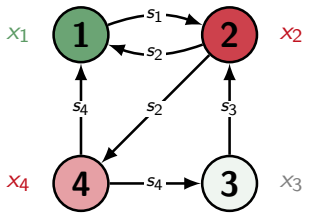
$$\dot{x}_i = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} a_{ij} \underbrace{s(x_j)}_{\text{Signal}} - x_i$$

Dynamics: Collective dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{D}^{-1} \boldsymbol{A} \boldsymbol{s}(\boldsymbol{x}) - \boldsymbol{x}$$

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Dynamics: Collective dynamics

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Assumption:

- s is common.
- s is K-Lipschitz, nondecreasing.

Outline

1. Opinion Dynamics presentation

2. Reaching a Consensus
   - 2.1 The Synchronization Manifold and the Scalar Dynamics
   - 2.2 The Invariant Hypercubes
   - 2.3 Summing Up: What we know about Consensus?

3. Emergence of Polarization

4. Coupling Opinion and Environment

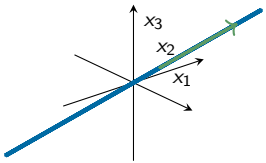
5. Emergence of Oscillations

6. Conclusion

The Synchronization Manifold

We define the **Fully Synchronized (FS) Manifold** $\mathcal{S}$:

$$\begin{aligned}\mathcal{S} &= \{\boldsymbol{x} \in [-1,1]^N \mid x_1 = x_2 = \cdots = x_N\} \\ &= \{c\mathbf{1} \mid c \in [-1,1]\}\end{aligned}$$

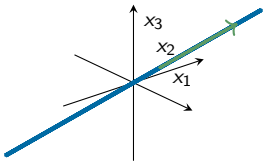


The complex network becomes a single particle moving on a line.

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The complex network becomes a single particle moving on a line.

Forward Invariance of $\mathcal{S}$

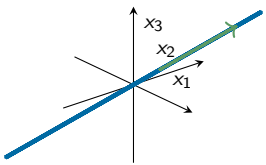
If agents start synchronized, they stay synchronized.

Reason: Let $\boldsymbol{x} = c\mathbf{1} \in \mathcal{S}$. Then, $\boldsymbol{s}(\boldsymbol{x}) = s(c)\mathbf{1}$. By Perron-Frobenius theorem, $\boldsymbol{D}^{-1}\boldsymbol{A}\mathbf{1} = \mathbf{1}$.
Thus, $\dot{\boldsymbol{x}} = \boldsymbol{D}^{-1}\boldsymbol{A}s(c)\mathbf{1} - c\mathbf{1} = (s(c) - c)\mathbf{1}$.

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The complex network becomes a single particle moving on a line.

Forward Invariance of $\mathcal{S}$

If agents start synchronized, they stay synchronized.

Dynamics on $\mathcal{S}$: If $\boldsymbol{x}(t) = c(t)\mathbf{1}$, the network term vanishes ($\sum_j a_{ij} = d_i$). The system reduces to a **scalar ODE**:

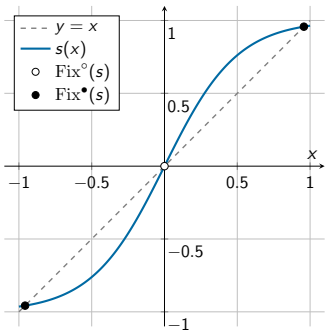
$$\dot{c} = s(c) - c$$

1D Dynamics: The Role of the Signal

The scalar dynamics $\dot{c} = s(c) - c$ is determined by the shape of s.

- The equilibria of the scalar dynamics are the **fixed points** of the signal function $s(x)$.

$$\mathrm{Fix}(s) = \mathrm{Fix}^{\bullet}(s) \cup \mathrm{Fix}^{\circ}(s).$$



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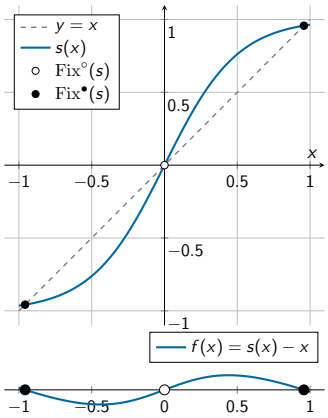
- The equilibria of the scalar dynamics are the **fixed points** of the signal function $s(x)$.

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- The attraction and stability depend on the **slope of $s(x)$ around the equilibria**.

  Stable eq.: s must cross the identity line $y = x$ entering from above and exiting from below.

  Proof: Take the Lyapunov function $V(c) = (x - c)^2$ and compute its time derivative.



1D Dynamics: The Role of the Signal

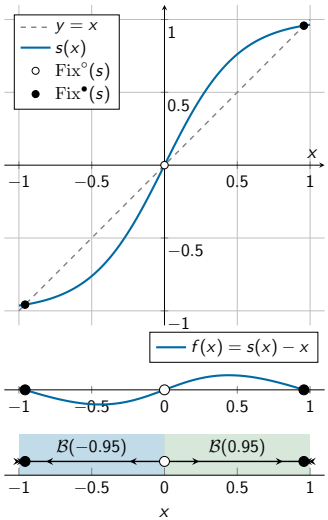
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- The basins of attraction are the delimited regions by unstable fixed points.



1D Dynamics: The Role of the Signal

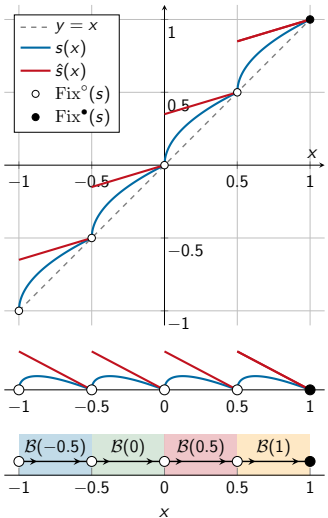
The scalar dynamics $\dot{c} = s(c) - c$ is determined by the shape of s.

- The equilibria of the scalar dynamics are the **fixed points** of the signal function $s(x)$.

$$\mathrm{Fix}(s) = \mathrm{Fix}^{\bullet}(s) \cup \mathrm{Fix}^{\circ}(s).$$

- The attraction and stability depend on the **slope of $s(x)$ around the equilibria**.
  Stable eq.: s must cross the identity line $y = x$ entering from above and exiting from below.

- The basins of attraction are the delimited regions by unstable fixed points.



Global Structure: Invariant Hypercubes

How does the 1D analysis inform the N-dimensional system?

Invariant Hypercubes

If c_1 and c_2 are two fixed points of $s(x)$, the hypercube

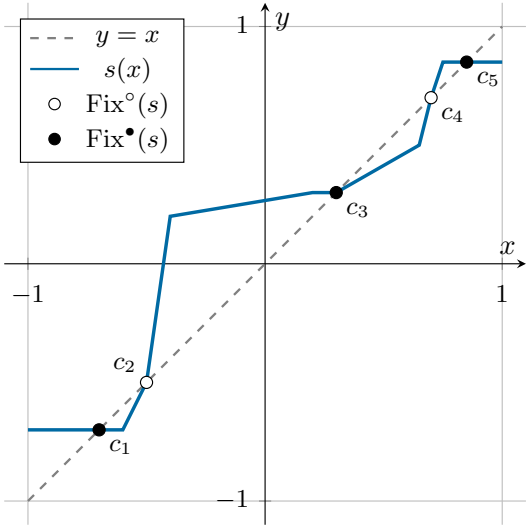
$$\mathcal{H} = [c_1, c_2]^N$$

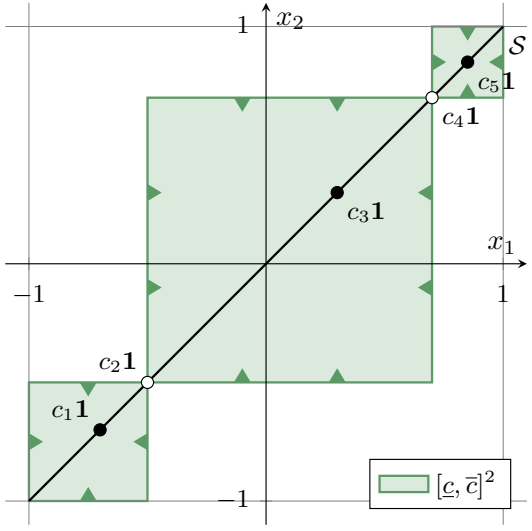
is **forward invariant**.

Proof: Using Nagumo's theorem (see, [Blanchini, 1999]).

- Trajectories are trapped in boxes defined by the signal s.
- Agents cannot cross all together the fixed points.
- **Implication:** Global behavior is partitioned into independent regions.

Induced Invariant Hypercubes generated by the signal s





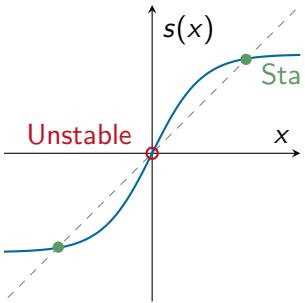
Summing Up: What we know about Consensus?

Fully Synchronized Equilibria (FSE):

- All agents agree: $\boldsymbol{x}^* = c\mathbf{1}$.
- Synchronization points $\Leftrightarrow$ Fixed points of s.

Stability Condition "simplified":

- Local stability depends on slope $s'(c)$.
- Stable if $|s'(c)| < 1$ (Under-reaction).
- Unstable if $|s'(c)| > 1$ (Over-reaction).
- Attraction Basins: Are (at least) the induced hypercubes by the signal s.



"Stable consensus is only possible at stable fixed points"

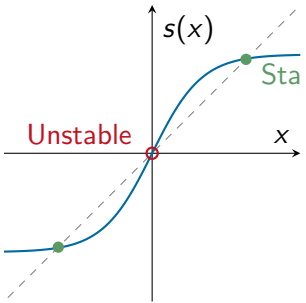
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What about polarization?

Outline

1. Opinion Dynamics presentation

2. Reaching a Consensus

3. Emergence of Polarization
   - 3.1 A Constructive Example
   - 3.2 Necessary Conditions on the Signal for Polarization
   - 3.3 A Sharp Threshold for Synchronization
   - 3.4 Global Convergence: Eventually vs. Directly
   - 3.5 Summing Up: The Recipe for Polarization

4. Coupling Opinion and Environment

5. Emergence of Oscillations

Emergence of Polarization: A Constructive Example

Setup:

- Line graph with $N = 5$ agents.
- Sigmoidal signal $s(x) = \tanh(Kx)$.
- **Parameter K:** Gain / Steepness.



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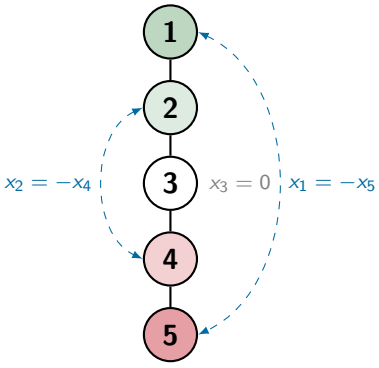
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Symmetry Reduction

The system admits an invariant subspace $\mathcal{M}$:

- Center is neutral: $x_3 = 0$.
- Symmetry: $x_1 = -x_5$ and $x_2 = -x_4$.

Dynamics reduce from 5D $\rightarrow$ 2D.



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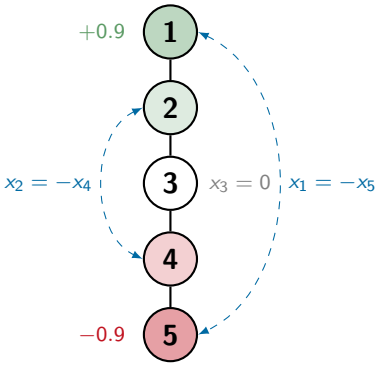
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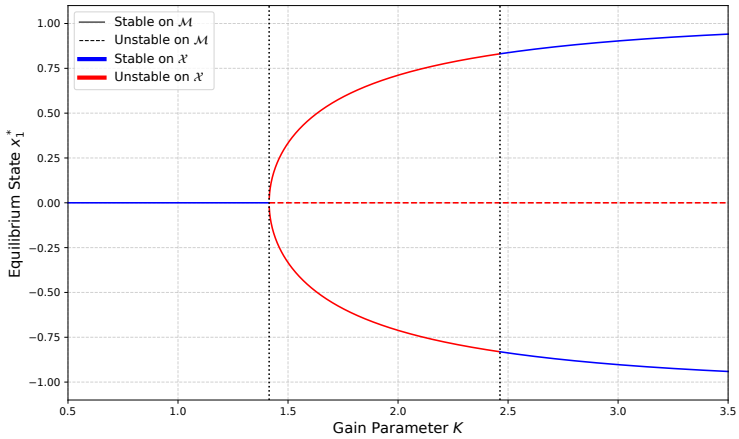
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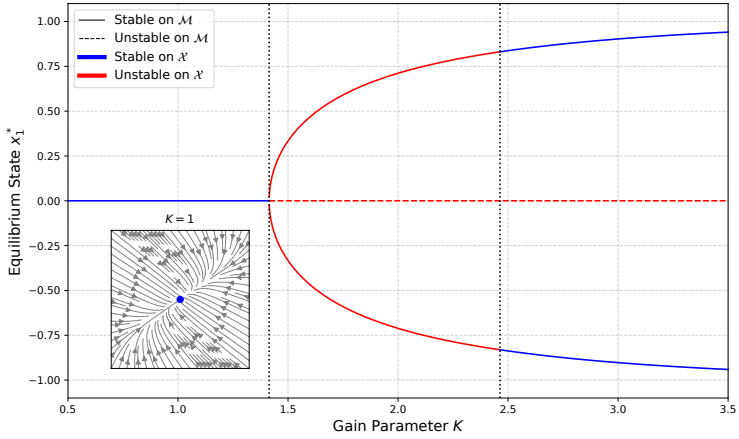
Breaking Consensus: The Bifurcation

How does the equilibrium x^* change as we increase nonlinearity K?



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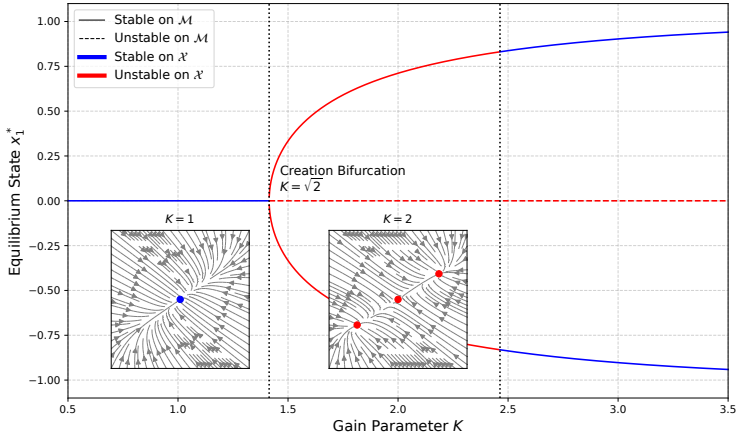
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- **Low K:** Only consensus ($x^* = 0$) exists and is stable.

Breaking Consensus: The Bifurcation

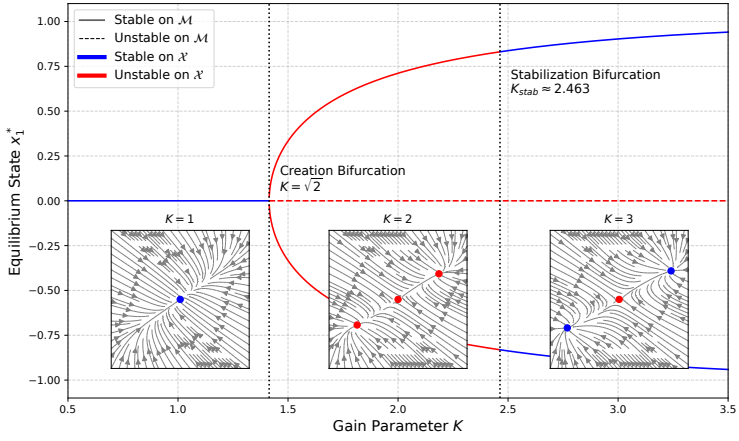
How does the equilibrium x^* change as we increase nonlinearity K?



- **Creation ($K > \sqrt{2}$):** Polarization emerges but is *saddle-unstable*.

Breaking Consensus: The Bifurcation

How does the equilibrium x^* change as we increase nonlinearity K?



- **Stabilization ($K > K_{\text{stab}}$):** Polarization becomes **locally stable**.

Necessary Conditions for Polarization

Theorem [Couthures et al., 2025]

For a persistent disagreement state $\boldsymbol{x}^*$ to exist:

1. **Instability Source:** The signal s must have at least one **unstable fixed point** c°.

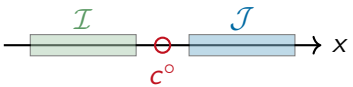
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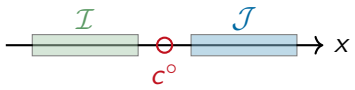
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3. **Topological Constraint:** Both groups must be significant.

$$|\mathcal{I}| \geq 2 \quad \text{and} \quad |\mathcal{J}| \geq 2$$



Implication:
Graphs with $N < 4$ or Star graphs **cannot** support polarization.

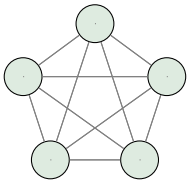
The Critical Role of Network Topology

Algebraic Connectivity λ_{N-1}

Eigenvalues of $\boldsymbol{D}^{-1}\boldsymbol{A}$: $-1 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} < \lambda_N = 1$.

- $\lambda_{N-1} \approx 1 \implies$ Sparse / String-like (Weak consensus force).

Highly Connected Graphs ($\lambda_{N-1} \leq 0$)



ALWAYS Synchronize

(Lemma: Disagreement structurally impossible)

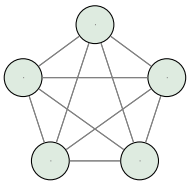
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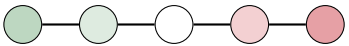


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Weakly Connected Graphs

($\lambda_{N-1} > 0$, e.g., Line Graph)



Can Polarize

(If signal is steep enough)

A Sharp Threshold for Synchronization

Can we predict the outcome by looking at K and λ_{N-1}?

Synchronization Guarantee [Couthures et al., 2025]

If $K\lambda_{N-1} < 1$, then the only stable equilibria are synchronized states.

Proof: Using Ostrowski's inequality (see, [Horn, 2012]).

- **Sufficient Condition:** If met, NO polarization is possible, regardless of $s(x)$ shape.
- **Sharpness:** If $K\lambda_{N-1} \geq 1$, there *exists* a signal that creates polarization.

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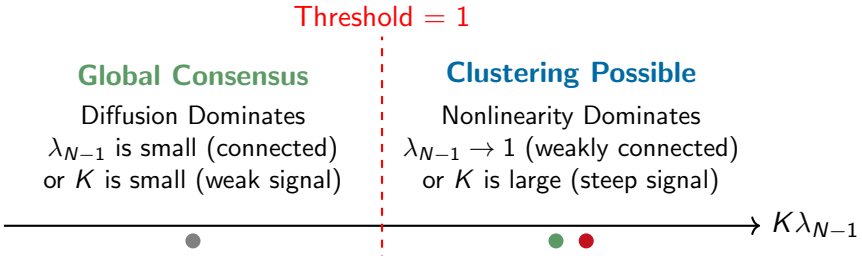
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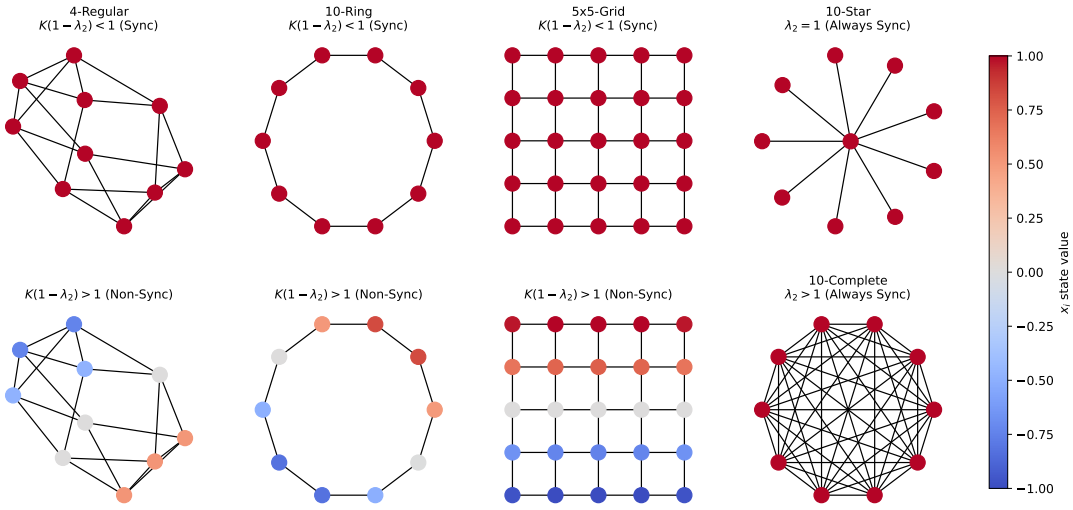
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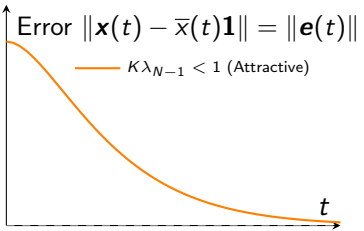


Synchronization vs. No Synchronization



Global Convergence: Eventually vs. Directly

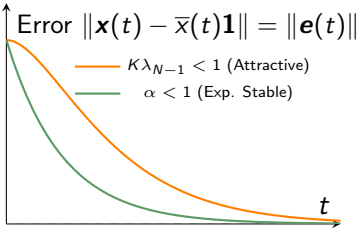
1. **Global Attractivity (The "Eventually")**
   - **Result:** If $K\lambda_{N-1} < 1$, all trajectories converge to the synchronization manifold $\mathcal{S}$.
   - **Caveat:** Convergence might not be monotonic.
   - *Transient growth* of disagreement is possible!



Global Convergence: Eventually vs. Directly

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   - **Caveat:** Convergence might not be monotonic.

2. **Exponential Stability (The "Directly")**
   - To guarantee fast, monotonic decay, we need a stricter condition on the **entire spectrum**.



Why the difference?

Negative eigenvalues (e.g., bipartite) can cause transient error growth.

Global Exponential Stability Condition [Couthures et al., 2025]

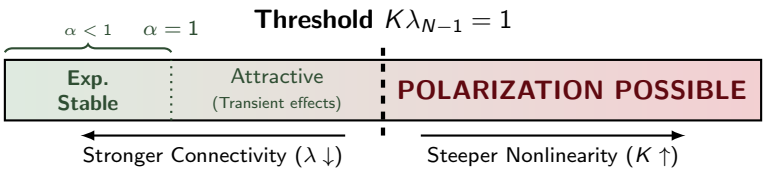
Let $\alpha := K \max_{i<N} |\lambda_i|$. If $\alpha < 1$, the manifold $\mathcal{S}$ is **Globally Exponentially Stable**.

Proof: Using Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{e}^\top \boldsymbol{D}^{-1} \boldsymbol{e}$ and Cauchy-Schwarz inequality.

Summing Up: The Recipe for Polarization

1. **The Mechanism (Signal)**
   - Polarization requires a **"Splitting Point"**.
   - There must exist an unstable fixed point c° in the signal $s(x)$.
   - Agents to the left of c° are pushed down; agents to the right are pushed up.

2. **The Condition (Network)**
   - Signal shape is not enough.
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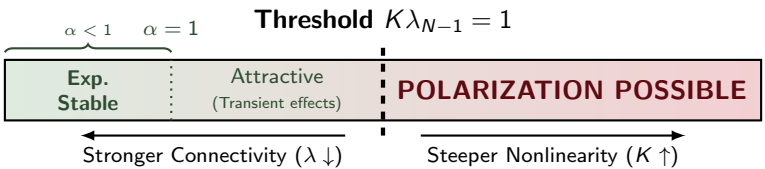
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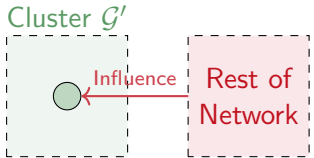


Polarization is possible... but is it useful?

From Global Existence to Local Robustness

The Challenge:

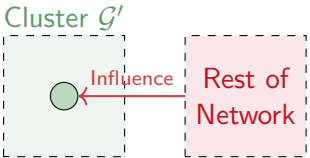
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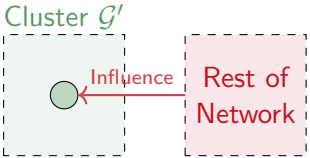
What is a "Cluster" here?
It is not just a graph cut but a **dynamical unit**:

- Consider the subgraph induced by $\mathcal{V}'$: $\mathcal{G}'$.
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Internal Stability Condition

$$K\lambda^{\text{in}}_{N'-1} < 1$$

Robustness: The Input-to-State Stability (ISS) Result

- **Internal Stability:** $\alpha_{\text{in}} = 1 - K\lambda^{\text{in}}_{N'-1}$.
- **Perturbation $\boldsymbol{p}(t)$**: External influence from the rest of the network.
- **Residual $\tilde{\boldsymbol{p}}(t)$**: Deviation from the average.

$$\tilde{\boldsymbol{p}} = \boldsymbol{p} - \underbrace{\bar{p}\mathbf{1}}_{\text{Average}}$$

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ISS of the error $\boldsymbol{e}(t)$ in the cluster

If $\alpha_{\text{in}} > 0$, the internal disagreement $\boldsymbol{e}(t)$ is bounded by the **heterogeneity** of external inputs $\tilde{\boldsymbol{p}}$:

$$\limsup_{t\to\infty} \|\boldsymbol{e}(t)\| \leq \frac{1}{\alpha_{\text{in}}} \underbrace{\sup \|\tilde{\boldsymbol{p}}(t)\|}_{\text{Non-uniformity}}$$

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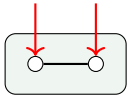
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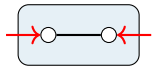
$\boldsymbol{p} = c\mathbf{1} \implies \tilde{\boldsymbol{p}} = \mathbf{0}$



Cluster moves together ($\boldsymbol{e} \to 0$)

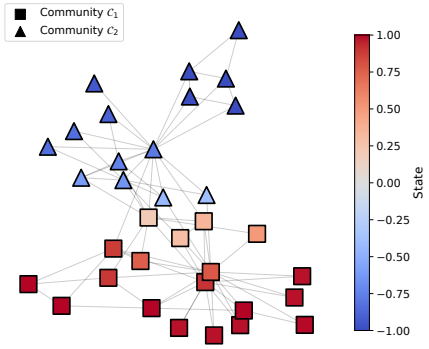
Case B: Heterogeneous Input

$\boldsymbol{p} \neq c\mathbf{1} \implies \tilde{\boldsymbol{p}} \neq \mathbf{0}$



Internal Disagreement $\boldsymbol{e} \neq 0$
(Bounded by α_{in})

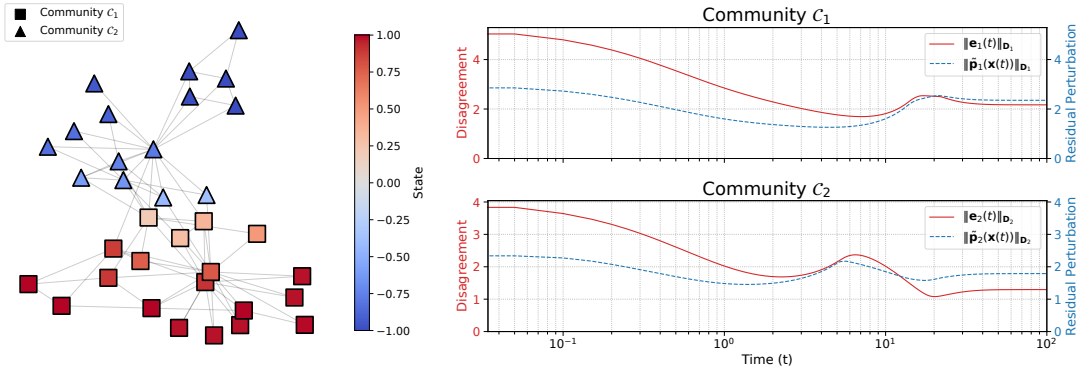
Community detection and ISS properties



(a) Karate Club network with 2 communities.

Figure: Local synchronization over the Karate Club network with an ISS property depending of perturbation from other communities.

Community detection and ISS properties



(a) Karate Club network with 2 communities.

(b) ISS analysis of the 2 communities.

Figure: Local synchronization over the Karate Club network with an ISS property depending of perturbation from other communities.

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3. Emergence of Polarization

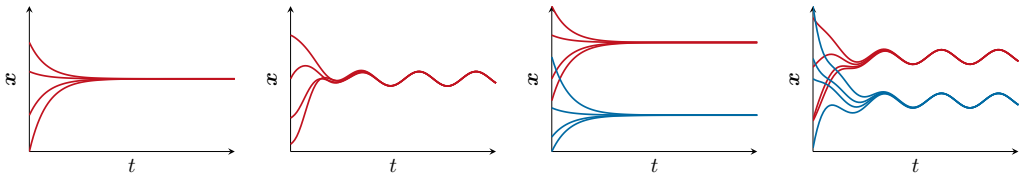
4. Coupling Opinion and Environment
   - 4.1 Environmental Dynamics
   - 4.2 Coupling Opinion and Environment

5. Emergence of Oscillations

6. Conclusion

Understand the evolution of opinions

What do we observe?



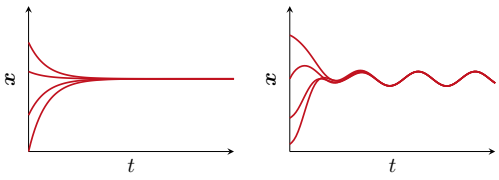
(a) Constant consensus (b) Oscillating consensus (c) Constant polarized (d) Oscillating polarized

Questions:

1. How do we shift from consensus to polarization?
2. How do we shift from constant consensus to oscillating consensus?

Understand the evolution of opinions

What do we observe?



(a) Constant consensus (b) Oscillating consensus

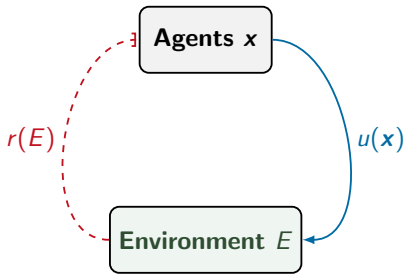
Questions:

2. How do we shift from constant consensus to oscillating consensus?

Environmental Dynamics

- **Shared Resource:** Common state E.
- **Collective Impact:**
  - Agents exert influence $u(\boldsymbol{x})$.
  - $u(\boldsymbol{x}) \in [u_{\min}, u_{\max}]$.
- **Dynamics:** (Time scale τ_e)

$$\tau_e \dot{E} = -\gamma E + u(\boldsymbol{x})$$



Closed-loop system

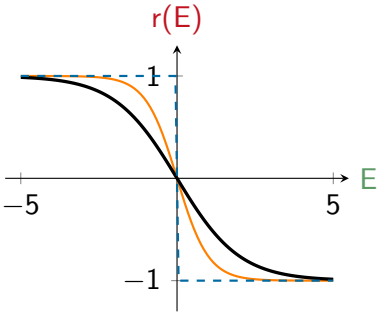
Simplified model inspired by [Berkowicz, 2000], [Garg, 2021].

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- **Feedback Signal** $r(E)$:
  - Decreasing function $[-1, 1]$.
  - **Restoring force:**
  - If $E > 0$ (Pollution) $\rightarrow r < 0$ (Discourage).



High state $e \rightarrow$ Negative feedback

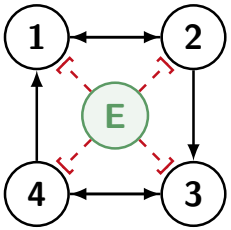
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Coupling Opinion and Environment

- **Dynamics:** Opinion $\dot{x}_i$ is driven by a mix of forces.

$$\tau_x \dot{x}_i = -x_i + \underbrace{\beta r(E)}_{\text{Env. Feedback}} + \underbrace{(1-\beta)\frac{1}{d_i}\sum a_{ij} s(x_j)}_{\text{Social Influence}}$$

- τ_x: Opinion time constant.
- $\beta \in [0, 1]$: Attention parameter.



Star Topology Interaction
E acts as a global node connected to everyone.

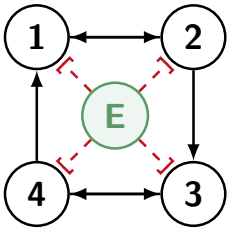
Assumption: Uniform influence ($\beta_i = \beta$) for all agents.

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Star Topology Interaction
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Trade-off Mechanism

- $\beta \to 0$: **Social Bubble** (Echo chamber).
- $\beta \to 1$: **Rational/Scientific** (Follows feedback).

Assumption: Uniform influence ($\beta_i = \beta$) for all agents.

Outline

1. Opinion Dynamics presentation

2. Reaching a Consensus

3. Emergence of Polarization

4. Coupling Opinion and Environment

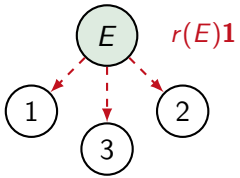
5. Emergence of Oscillations
   - 5.1 Model reduction
   - 5.2 Mathematical Mechanism: Pitchfork and Hopf Bifurcations
   - 5.3 The Global Picture: Bifurcation Diagram

6. Conclusion

Model Reduction: The Power of Homogeneity

Assume we start on the **Fully Synchronized Opinion (FSO)** manifold:

$$\mathcal{S} = \{(\boldsymbol{x}, E) \mid \boldsymbol{x} = p\mathbf{1}\}$$



Since E pushes everyone equally, no disagreement is created.

Model Reduction: The Power of Homogeneity

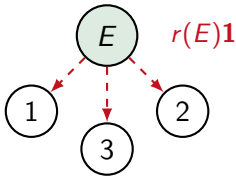
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Reason: Why do we stay on the FSO?

$$\tau_x \dot{\boldsymbol{x}} = \underbrace{-\boldsymbol{L}_s(\boldsymbol{x})}_{\text{Consensus}} + \underbrace{\beta r(E)\mathbf{1}}_{\text{Environmental Input}}$$

- The consensus term preserves synchronization.
- The environment acts as a **Uniform Perturbation** ($\tilde{\boldsymbol{p}} = 0$).



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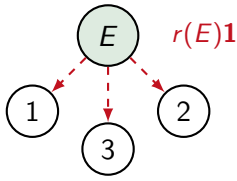
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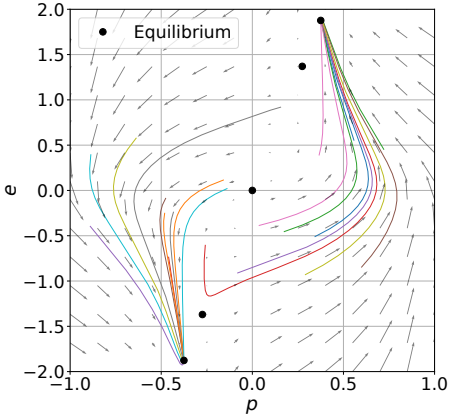
Reduced Planar System

Since $\boldsymbol{x}(t)$ remains of the form $p(t)\mathbf{1}$, the N equations collapse to **one**:

$$\begin{cases} \tau_x \dot{p} = -p + \beta r(E) + (1-\beta)s(p) \\ \tau_e \dot{E} = -\gamma E + u(p) - \gamma \bar{E} \end{cases}$$

Observing the Phenomenon

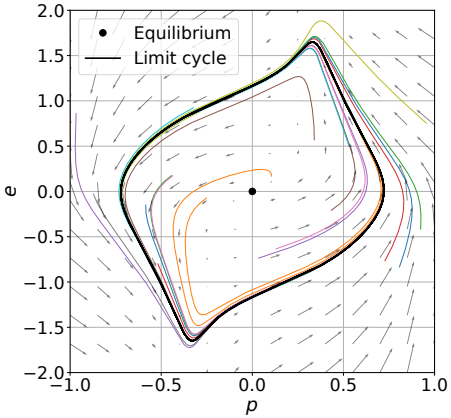
What happens when we vary the attention to the environment (β)?



1. Low β ($\beta = 0.24$): Convergence to Equilibria (Bistability).

Observing the Phenomenon

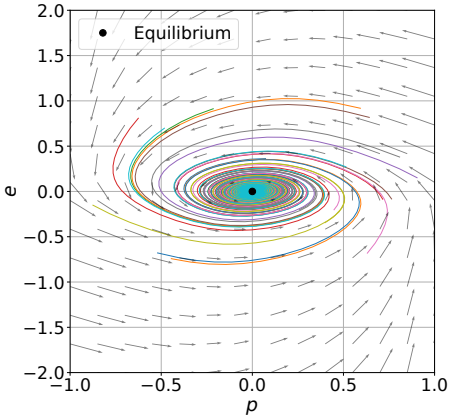
What happens when we vary the attention to the environment (β)?



2. Critical β ($\beta = 0.25$): Oscillations Emerge (Limit Cycle).

Observing the Phenomenon

What happens when we vary the attention to the environment (β)?



3. High β ($\beta = 0.61$): Return to Equilibrium (Global Consensus).

Mathematical Mechanism: Pitchfork and Hopf Bifurcations

How mathematically? We analyze the Jacobian $\boldsymbol{J}$ of the 2D system around $(0, 0)$.

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Mechanism 1: Symmetry Breaking

Pitchfork Bifurcation

- **Condition:** One eigenvalue $\lambda = 0$.
- **Cause:** Opinion Amplification.

$$(1 - \beta)s'(0) > 1$$



Creates polarized Pro/Anti states.

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Mechanism 2: Cyclic Behavior

Hopf Bifurcation

- **Condition:** Eigenvalues $\lambda = \pm i\omega$.
- **Cause:** Frustration.

$$\text{Trace}(\boldsymbol{J}) = 0$$

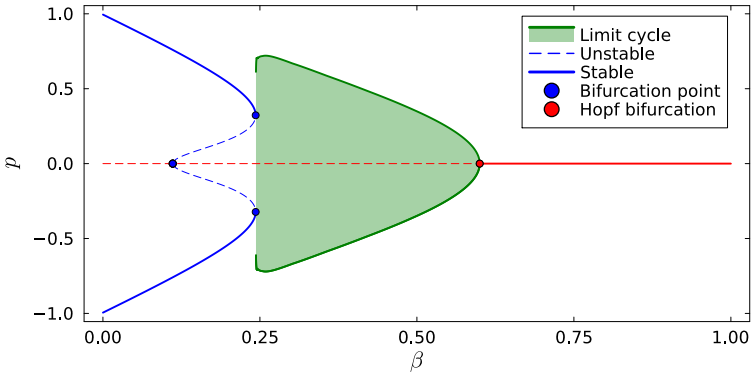


Creates predator-prey oscillations.

Proofs: Lyapunov-Schmidt reduction on point making the Jacobian singular. Transversality condition on the eigenvalues.

The Global Picture: Bifurcation Diagram

Summarizing all behaviors in a single **Bifurcation Diagram**.



Regimes:

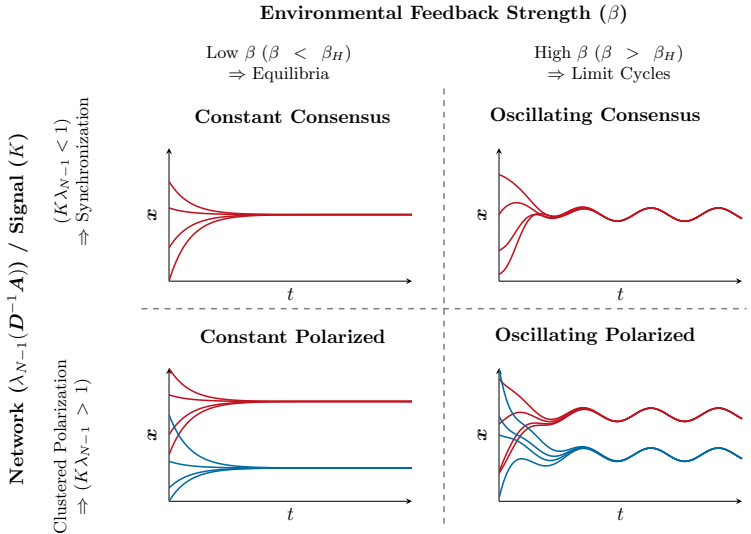
1. **Blue Lines:** Polarization.
2. **Green Area:** Oscillations.
3. **Red Line:** Neutral State.

Key Takeaway: Low trust (β) leads to Polarization. Intermediate trust leads to Oscillations. High trust restores Neutrality.

Outline

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The Global Picture: Full Picture



Conclusion & Perspectives

1. **From Consensus to Polarization:**
   - Established a **Sharp Threshold** $K\lambda_{N-1} < 1$.
   - Polarization requires **amplification** (signal) and **weak connectivity** (graph).
2. **Robustness of Clusters:**
   - Proved cluster stability using **ISS**.
   - Key Insight: Internal cohesion resists **heterogeneous** external disturbances.
3. **Environmental Coupling:**
   - Feedback loops create complex dynamics (**Oscillations/Hopf**).

- **Heterogeneity:** What if agents have different sensitivities (β_i) or signals (s_i)?
- **Network Control:** How to optimally rewire the graph or inject signals to break polarization?
- **Global Bifurcation:** Extending the oscillation analysis beyond the invariant manifold (Full N-dim analysis).