Influence des dynamiques d'opinion sur le changement climatique : analyse et stratégies macroscopiques

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CRAN - UMR 7039 - Université de Lorraine

École Doctorale IAEM

Mention : Automatique, Traitement du Signal et des Images

8 Avril 2026



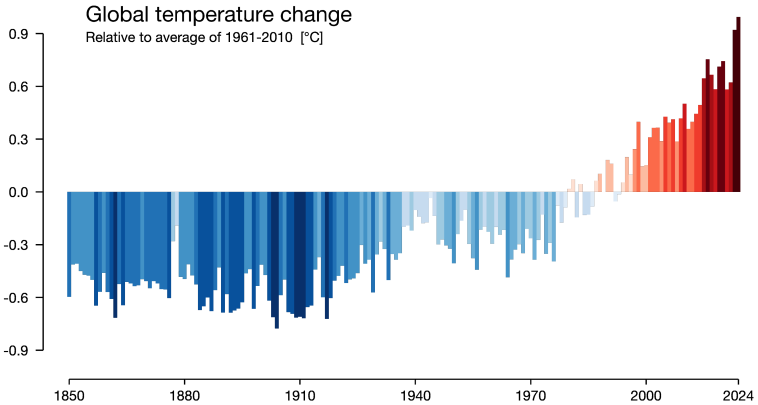




Context: The Climate Crisis Paradox

The Climate Crisis:

- CO2 accumulation and temperature are rising and the trend is accelerating.
- Bad for human health (desertification, heatwaves, etc.) and economy (agriculture, tourism, etc.).



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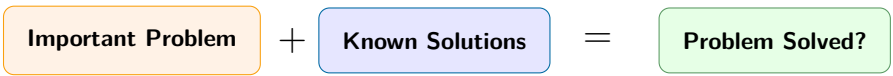
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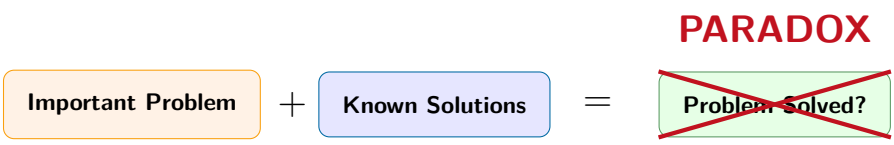
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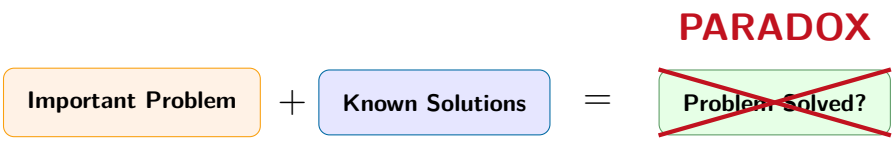
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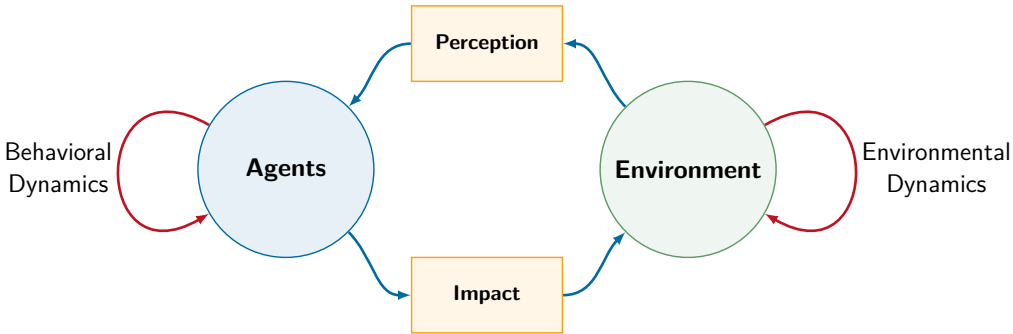


The Core Issue

This crisis is not only climate science problem, but also a **sociological one** that needs to be understood. **We lack the mathematical tools to understand it**.

How to model this socio-environmental system?

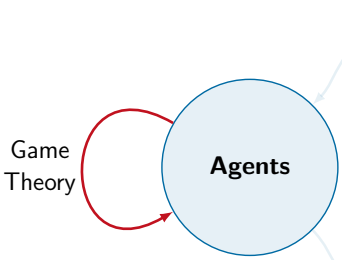
A mathematical sociological modeling problem:



Dynamics Agents have an **impact** on a **dynamic environment**, their behavior is influenced by their **perception** of the environment.

How to model this socio-environmental system?

A mathematical sociological modeling problem:



Two Modeling Approaches

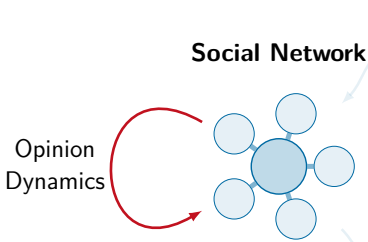
1. **Strategic**: Optimizers of their own utility.

Publications with Strategic Agents (Game theory):

- "A Quadratic Static Game Model for Assessing the Impact of Climate Change", Mroué, Couthures, et al. (2024)
- "On the Impact of Cross-Country Imitation on Climate Change", Mroué, Couthures, et al. (2025)
- "International Climate Policy as a Potential Game", Couthures et al. (sub2026)

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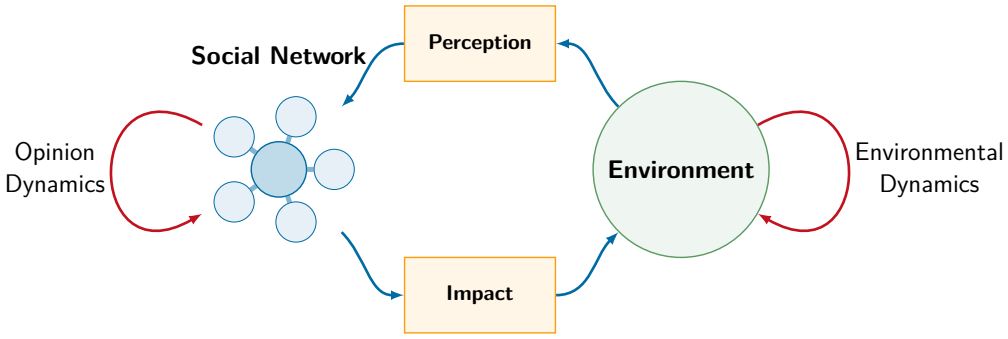
1. **Strategic**: Optimizers of their own utility.
2. **Non-strategic**: Evolving with social influence.

Publications with non-strategic agents (Opinion Dynamics with environment):

- "Analysis of a CODA Model Coupled with an External Dynamics", Couthures et al. (2024a)
- "Analysis of an OD Model Coupled with an External Environmental Dynamics", Couthures et al. (2024b)
- "Bifurcation Analysis of an OD Model Coupled with an Environmental Dynamics", Couthures et al. (2025a)

How to model this socio-environmental system?

A mathematical sociological modeling problem:



For this talk, we will focus on **understanding the behavior of a group of agents interacting with an environment** in the context of **opinion dynamics on graphs**.

Our Model

- **Opinion Dynamics:** for agent i, the opinion dynamics is given by:

$$\dot{x}_i = -x_i + \underbrace{\beta r(E)}_{\text{Env. Feedback}} + \underbrace{(1-\beta)\frac{1}{d_i}\sum_{j\in\mathcal{V}} a_{ij}s(x_j)}_{\text{Network Social Influence}}$$

- $\beta \in [0,1]$: Attention parameter.
- $x_i \in [-1,1]$: Agent i opinion.

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Two main improvements over the literature

Only 1 agent vs the env.:

- (Weitz et al. 2016)
- (Tilman et al. 2020)
- (Ito and Yamamichi 2024)

Polarization on static graphs need signed interaction:

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→ Naomi Leonard's group

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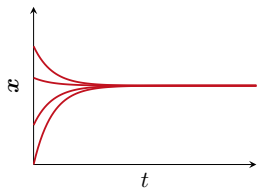
- **Environmental dynamics:** Physical process driving the evolution of the environment.

$$\tau_e \dot{E} = -\gamma E + u(\boldsymbol{x})$$

  - τ_e: time scale.
  - $u(\boldsymbol{x})$: Agents' collective impact.
  - γ: Restoring force of the environment.

Modeling the evolution of opinions

The behaviors our model can capture are:



(a) Constant consensus

(b) Constant polarized

(c) Oscillating consensus

(d) Oscillating polarized

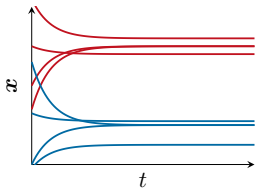
Captured behaviors: Constant consensus

Example: Scientific consensus on the existence of global warming.

Modeling the evolution of opinions

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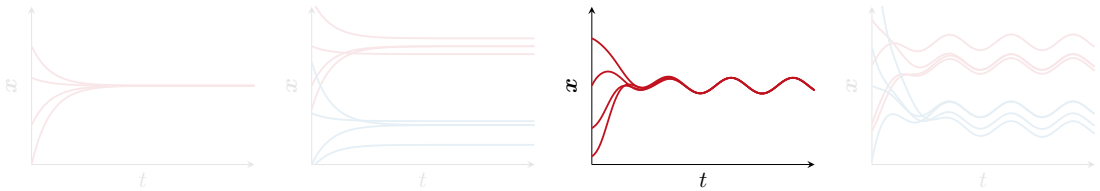
(d) Oscillating polarized

Captured behaviors: Polarized behavior

Example: Entrenched public division regarding nuclear energy or carbon taxation.

Modeling the evolution of opinions

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Captured behaviors: Oscillating consensus

Example: Collective risk perception fluctuating directly with environmental crises.

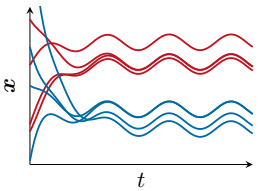
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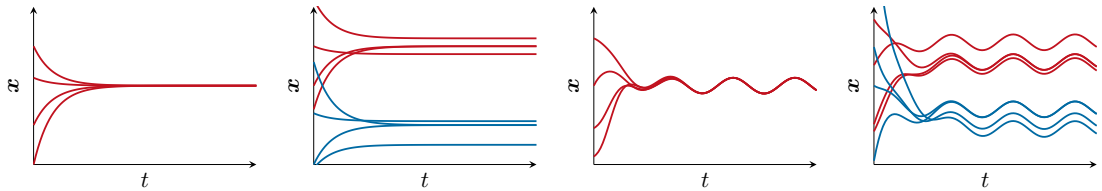
(d) Oscillating polarized

Captured behaviors: Oscillating polarized

Example: Partisan divides where the intensity fluctuates with political cycles or events.

Modeling the evolution of opinions

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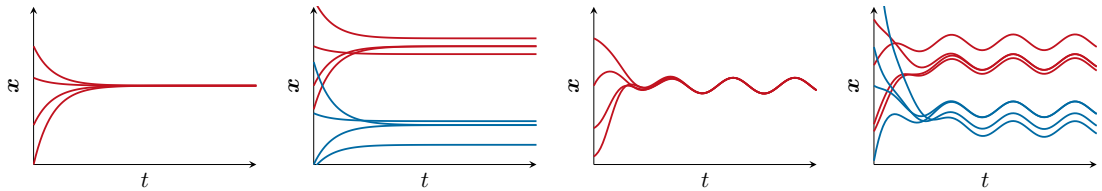
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Problems: The model is complex.

- The model is **nonlinear**.
- Dependent on **parameters**.
- In the early stages, the **nonlinearity was discontinuous**.
- Dependent on **graph structure**.
- Difficult to **analyze mathematically**.
- In discrete time, the system is **chaotic**.

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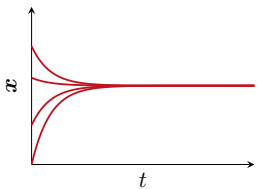
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Questions: How do the different behaviors emerge?

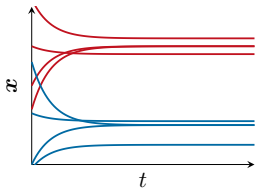
1. How do we shift **from consensus to polarization**?
2. How do we shift **from constant to oscillating consensus**?

Modeling the evolution of opinions

The behaviors our model can capture are:



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Questions: How do the different behaviors emerge?

1. How do we shift **from consensus to polarization**? The case where $\beta = 0$.
2. How do we shift **from constant to oscillating consensus**?

Outline

1. Opinion Dynamics

2. Consensus Characterization and Analysis

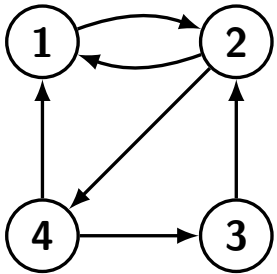
3. Emergence of Polarization and Clusters

4. Coupling Opinion and Environment

5. Conclusion

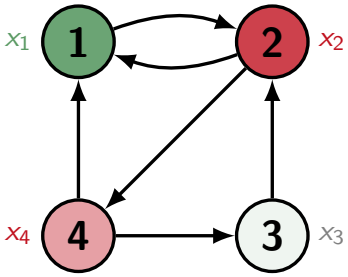
Basic Definition and Notation of Opinion Dynamics

- **Network:** N agents in $\mathcal{V} = \{1, 2, \ldots, N\}$.
- **Classic graph definition:**
  - **Adjacency** matrix $\boldsymbol{A}$ and **Degree** matrix $\boldsymbol{D}$
  - **Normalized adjacency** matrix $\boldsymbol{D}^{-1}\boldsymbol{A}$



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  - **Normalized adjacency** matrix $\boldsymbol{D}^{-1}\boldsymbol{A}$
- **Opinion State:** Each node i holds a value x_i.
- **Normalized Opinion Space** $x_i \in [-1, 1]$:
  - $+\mathbf{1}$: Strongly Favorable (Green).
  - $-\mathbf{1}$: Strongly Against (Red).
- **State Vector** $\boldsymbol{x} = (x_i)_{i \in \mathcal{V}}$.

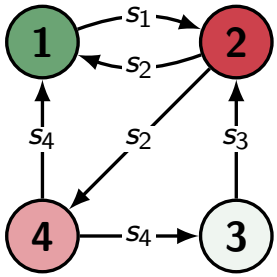


State Vector $\boldsymbol{x}$

$$\boldsymbol{x} = \begin{pmatrix} 0.9 \\ -0.8 \\ 0.1 \\ -0.4 \end{pmatrix} \begin{matrix} \leftarrow \text{Node 1} \\ \leftarrow \text{Node 2} \\ \leftarrow \text{Node 3} \\ \leftarrow \text{Node 4} \end{matrix}$$

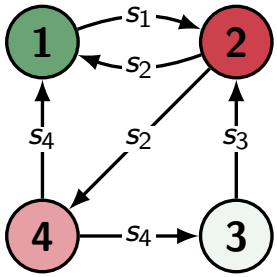
Opinion Dynamics with Nonlinear Communication

- Agents observe **signals** not opinions since opinions are private.
- **Signal** function $s : [-1, 1] \to [-1, 1]$.



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Dynamics Per-agent:

$$\dot{x}_i = \frac{1}{d_i} \sum_{j=1}^{N} a_{ij} \underbrace{s(x_j)}_{\text{Signal}} - x_i$$

Collective dynamics:

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Opinion Dynamics with Nonlinear Communication

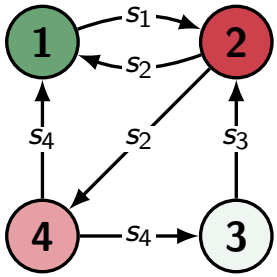
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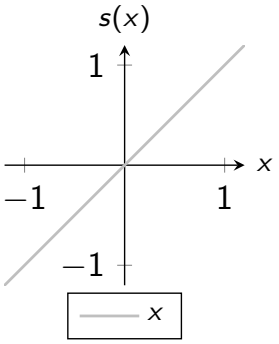
- The signal s is **common** and K-**Lipschitz**, **nondecreasing**.
- **Fixed** graph **symmetric** and **positive**.

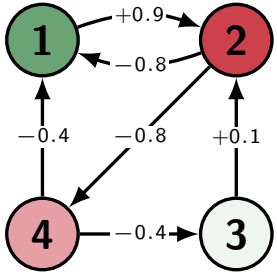
Proposition (sTAC-2025)

This system is cooperative and **no attracting limit cycle exists** $\Rightarrow$ **only equilibria***.

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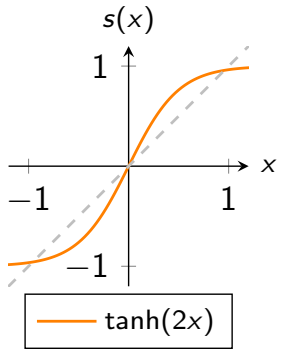
This is linear averaging.

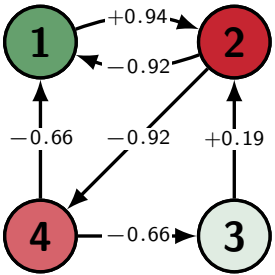
References:

- (French 1956) fondamental reference from mathematical sociology.
- (Degroot 1974) reach consensus

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Influence is smooth sigmoidal.

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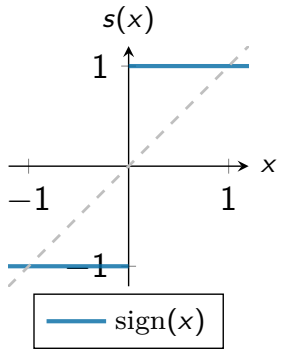
- (Baumann et al. 2020) Numerical,

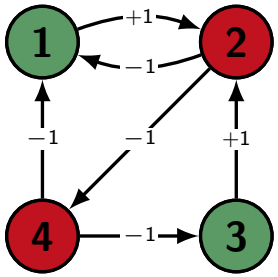
For signed graphs:

- (Gray et al. 2018),
- (Fontan and Altafini 2018), local analysis
- (Bizyaeva et al. 2023), local analysis

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Signal is quantized: discrete values.

References:

- (Martins 2008), CODA not an ODE,
- (Frasca 2012), Quantized OD,
- (Chowdhury et al. 2016) CODA,
- (Ceragioli et al. 2025) COBA.

Opinion Dynamics with Nonlinear Communication

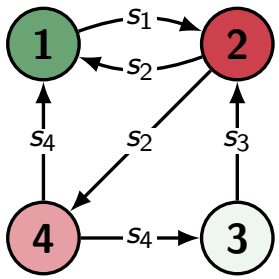
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Our model "unifies" those models.

References:

- (Couthures et al. 2025b),
- (Couthures et al. sub2025 TAC),
- (Couthures et al. sub2025 NAHS)

Outline

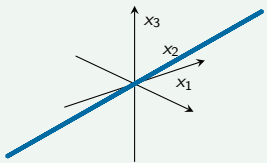
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2. Consensus Characterization and Analysis
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The Synchronization Manifold

Definition (LCSS-2025): Fully Synchronized Manifold

The **consensus manifold** is defined as $\mathcal{S}$:

$$\begin{aligned}\mathcal{S} &= \{\boldsymbol{x} \in [-1,1]^N \mid x_1 = x_2 = \cdots = x_N\} \\ &= \{c\mathbf{1} \mid c \in [-1,1]\}.\end{aligned}$$



Goal to understand the model:

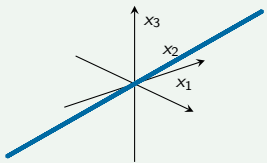
- What is happening on the synchronization manifold?
- Does every consensus is stable like in the linear case?
- What is the role of the signal s?

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Proposition (LCSS-2025): Forward Invariance of $\mathcal{S}$

If agents start synchronized, they stay synchronized.

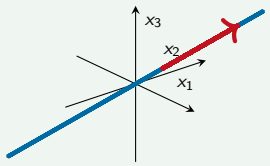
Proof: Let $\boldsymbol{x} = c\mathbf{1} \in \mathcal{S}$. Then, $\boldsymbol{s}(\boldsymbol{x}) = s(c)\mathbf{1}$. By Perron-Frobenius theorem, $\boldsymbol{D}^{-1}\boldsymbol{A}\mathbf{1} = \mathbf{1}$. Thus, $\dot{\boldsymbol{x}} = \boldsymbol{D}^{-1}\boldsymbol{A}s(c)\mathbf{1} - c\mathbf{1} = (s(c) - c)\mathbf{1}$.

The Synchronization Manifold

Definition (LCSS-2025): Fully Synchronized Manifold

The **consensus manifold** is defined as $\mathcal{S}$:

$$\begin{aligned}\mathcal{S} &= \{\boldsymbol{x} \in [-1,1]^N \mid x_1 = x_2 = \cdots = x_N\} \\ &= \{c\mathbf{1} \mid c \in [-1,1]\}.\end{aligned}$$



Proposition (LCSS-2025): Forward Invariance of $\mathcal{S}$

If agents start synchronized, they stay synchronized.

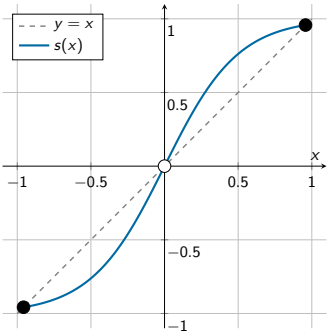
Dynamics on $\mathcal{S}$:
If $\boldsymbol{x}(t) = c(t)\mathbf{1}$, the network term vanishes ($\sum_j a_{ij} = d_i$). The system reduces to a **scalar ODE**:

$$\dot{c} = s(c) - c$$

1D Dynamics: The Role of the Signal

On the synchronization manifold $\mathcal{S}$:

- **Equilibria:** $\dot{c} = 0 \Leftrightarrow s(c) = c$. Intersections with identity line. i.e., fixed points: $\mathrm{Fix}(s)$.



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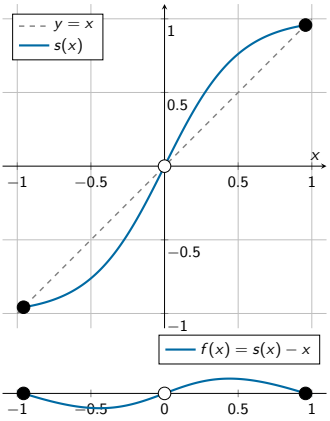
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Definition: Genre of fixed point

A fixed point $c \in \mathrm{Fix}(s)$ is:

- Stable if s crosses $y = x$ from **above** to **below**.
- Unstable if s crosses $y = x$ differently.

We partition $\mathrm{Fix}(s) = \underbrace{\mathrm{Fix}^{\bullet}(s)}_{\text{Stable}} \cup \underbrace{\mathrm{Fix}^{\circ}(s)}_{\text{Unstable}}$.



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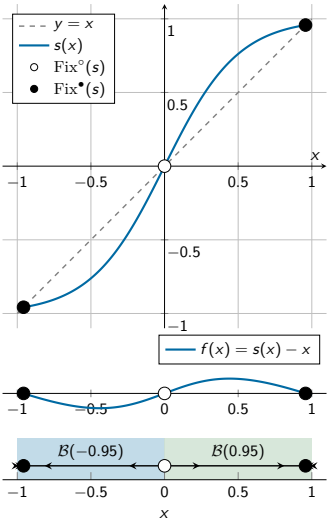
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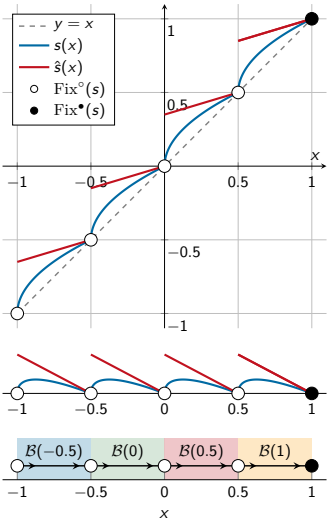
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Global Structure: Invariant Hypercubes

For the N-dimensional system, the 1D analysis informs the global structure of the system.

Proposition (sTAC-2025): Invariant Hypercubes

If c_1 and c_2 are two fixed points of $s(x)$, the hypercube

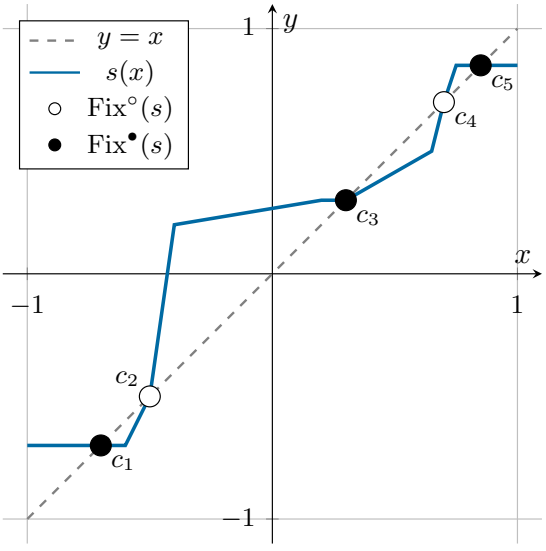
$$\mathcal{H} = [c_1, c_2]^N$$

is **forward invariant**.
Moreover, if $c_1^\circ < c^\bullet < c_2^\circ$ are consecutive fixed points, then $\mathcal{H}$ is in the **attraction basin** of $c^\bullet \mathbf{1}$.

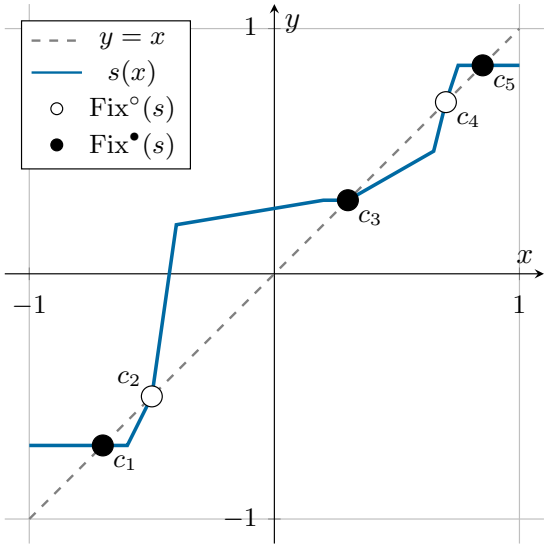
Proof: Using Nagumo's theorem (see, (Blanchini 1999)) and Lyapunov analysis arround the fixed points (from the row stochasticity of the adjacency matrix).

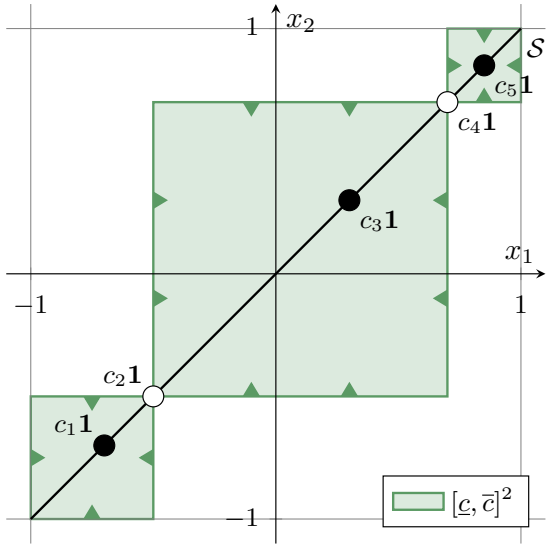
- Trajectories are **trapped in boxes** defined by the signal s.
- Agents cannot cross all together the fixed points.
- **Geometric Implication:** Global behavior is partitioned into independent regions.

Example for the 2-D system (intuition for the N-D system)



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Summing Up: What we know about Consensus?

Fully Synchronized Equilibria (FSE):

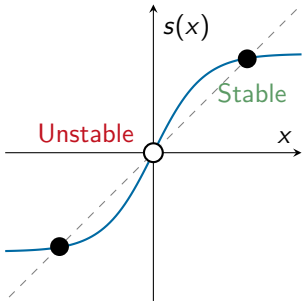
- All agents agree: $\boldsymbol{x}^* = c\mathbf{1}$.
- Synchronization points $\Leftrightarrow$ Fixed points of s.

Stability Condition "simplified":

- Local stability depends on slope $s'(c)$.
- Stable if $s'(c) < 1$ (Under-reaction).
- Unstable if $s'(c) > 1$ (Over-reaction).

Attraction Basins:

- Are (at least) the induced hypercubes by the signal s.



"Stable consensus is only possible at stable fixed points"

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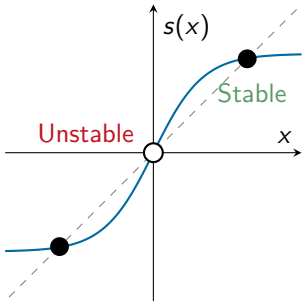
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"Stable consensus is only possible at stable fixed points"

We understand the behavior of the system **on and near the synchronization manifold** $\mathcal{S}$.

Outline

1. Opinion Dynamics

2. Consensus Characterization and Analysis

3. Emergence of Polarization and Clusters

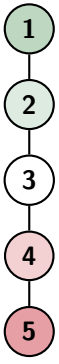
4. Coupling Opinion and Environment

5. Conclusion

Emergence of Polarization: A Constructive Example

Setup:

- Line graph with $N = 5$ agents.
- Sigmoidal signal $s(x) = \tanh(Kx)$.
- **Parameter K:** Gain / Steepness.



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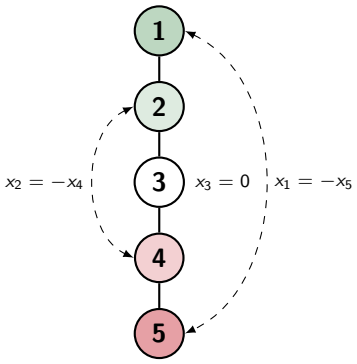
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Proposition (sTAC-2025): Symmetry Reduction

The system admits an invariant subspace $\mathcal{M}$:

- Center is neutral: $x_3 = 0$.
- Symmetry: $x_1 = -x_5$ and $x_2 = -x_4$.

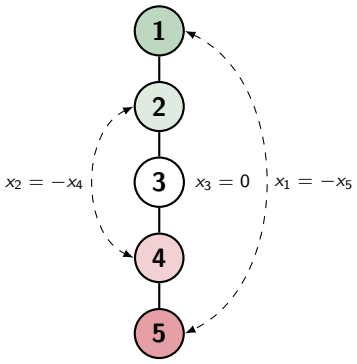
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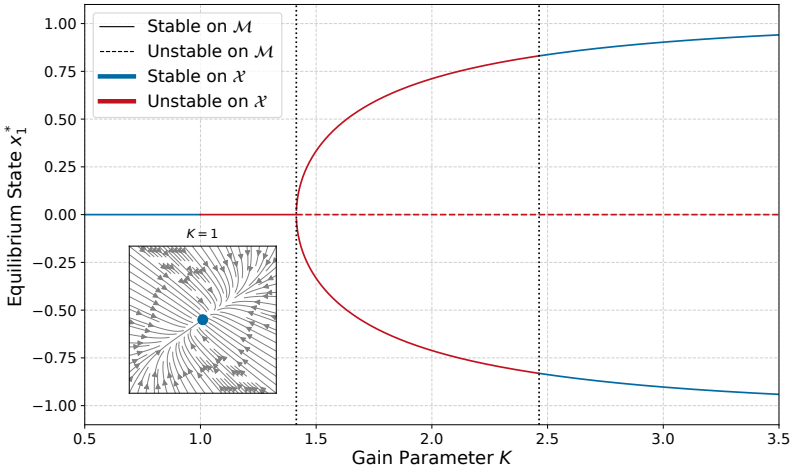
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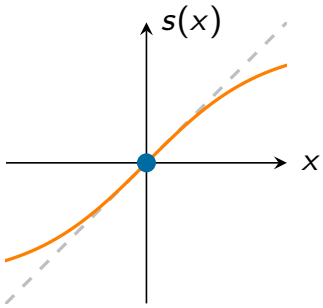
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Question:

How does the equilibrium x_1^* change as we increase nonlinearity K?

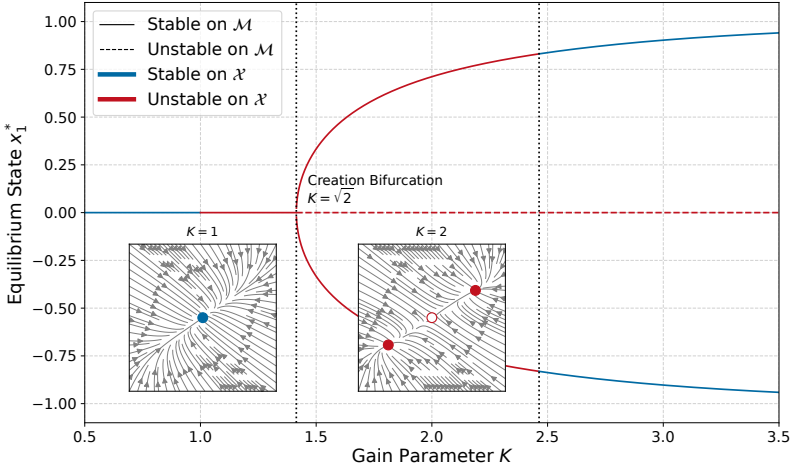
Breaking Consensus: The Bifurcation

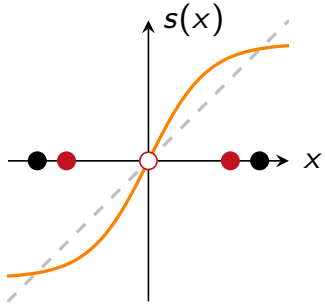




Low K: Only consensus ($x_1^* = 0$) exists and is stable.

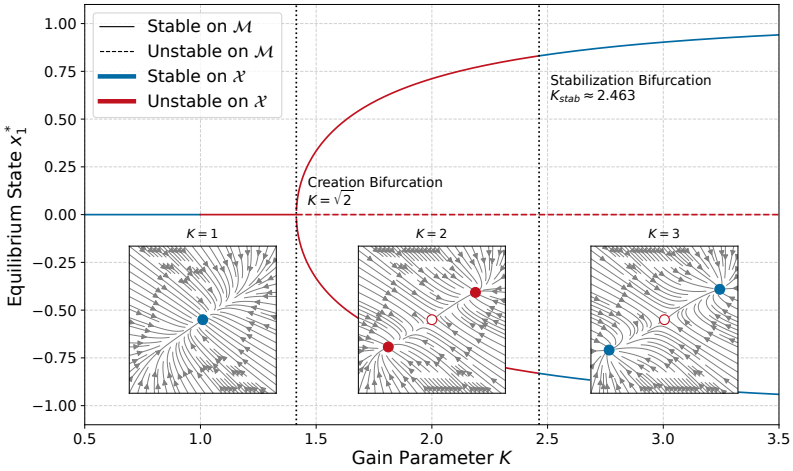
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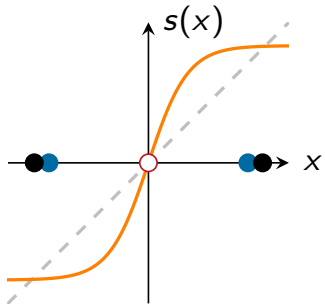




Creation ($K > \sqrt{2}$): Polarization emerges $x_1^* \notin \mathrm{Fix}^{\bullet}(s)$ but **stable w.r.t.** $\mathcal{M}$ but **unstable w.r.t.** $\mathcal{X}$ **(non-symetric perturbation)**.

Breaking Consensus: The Bifurcation





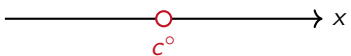
Stabilization ($K > K_{\text{stab}}$): Polarization becomes **locally stable w.r.t.** $\mathcal{X}$.

Necessary Conditions for Polarization

Theorem (sTAC-2025): Necessary Conditions for Polarization

For a polarized equilibrium state $\boldsymbol{x}^*$ to exist:

1. **Instability Source:** The signal s must have at least one **unstable fixed point** c°.



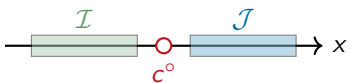
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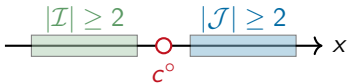
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Question:

What about the graphs with $N \geq 4$ and non-Star graphs?

The Critical Role of Network Topology

Definition: Algebraic Connectivity λ_{N-1}

Eigenvalues of $\boldsymbol{D}^{-1}\boldsymbol{A}$: $-1 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} < \lambda_N = 1$. We call λ_{N-1} the **algebraic connectivity**.

The Critical Role of Network Topology

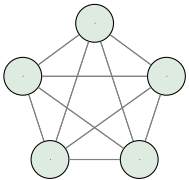
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- $\lambda_{N-1} \approx 0 \implies$ Dense / Complete (Strong consensus force).

Highly Connected Graphs

($\lambda_{N-1} \approx 0$, e.g., Complete Graph)



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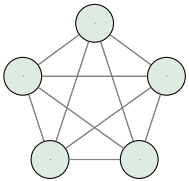
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Proposition (sTAC-2025)

For any graph such that $\lambda_{N-1} \approx 0$, then the system always synchronize.

Proposition (sNAHS-2025)

For complete bipartite graph, the system always synchronize.

The Critical Role of Network Topology

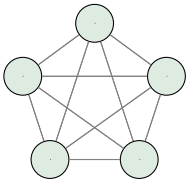
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- $\lambda_{N-1} \approx 1 \implies$ Sparse / String-like (Weak consensus force).

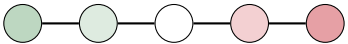
Highly Connected Graphs

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Weakly Connected Graphs

($\lambda_{N-1} \approx 1$, e.g., Line Graph)



A Sharp Threshold for Synchronization

Theorem (sTAC-2025): Synchronization Guarantee

If $K\lambda_{N-1} < 1$, then **the only equilibria are synchronized states**.

Proof: Using Ostrowski's inequality (see, (Horn and Johnson 2017)), similar to (Fontan and Altafini 2018) but generalizing it.

- **Sufficient Condition:** If met, **NO polarization is possible**, regardless of $s(x)$ shape.

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- **Sufficient Condition:** If met, **NO polarization is possible**, regardless of $s(x)$ shape.
- **Sharpness:** If $K\lambda_{N-1} \geq 1$, there *exists* a signal that creates polarization.

Proposition (sTAC-2025): Non-Synchronization Equilibrium

Let $K > 1$ and $N \geq 4$. For any graph $\mathcal{G}$, such that $\lambda_{N-1} > 0$. The following statements hold:

1. If $K\lambda_{N-1} = 1$, there exists s such that the system possesses a continuum of NFSE.
2. If $K\lambda_{N-1} > 1$, there exists s such that the system possesses NFSE.

Proof: Frontier of attraction bassin of hyperbolic system are stable manifolds of unstable equilibria (see, (Chiang and Alberto 2015)).

Error Dynamics and Global Convergence: Eventually vs. Directly

Definition: Error

- **Global Weighted Average:** $\bar{x}(t) = \frac{1}{N} \sum_{i=1}^{N} d_i x_i(t)$ for all t.
- **Error:** $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \bar{x}(t)\mathbf{1}$.

Global Attractivity (The "Eventually")

- **Condition:** If $K\lambda_{N-1} < 1$, $\|\boldsymbol{e}(t)\| \to 0$ as $t \to \infty$.

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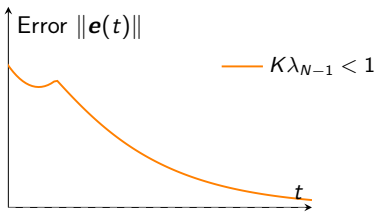
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- **Condition:** If $K\lambda_{N-1} < 1$, $\|\boldsymbol{e}(t)\| \to 0$ as $t \to \infty$.
- **But:** Convergence may be not monotonic.

Global Exponential Stability (The "Directly")

- **Condition:** Monotonic need the **entire spectrum**.



Error Dynamics and Global Convergence: Eventually vs. Directly

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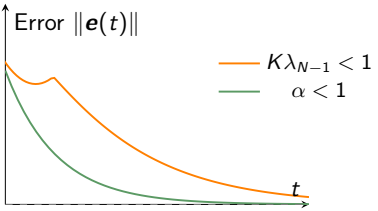
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Global Attractivity (The "Eventually")

- **Condition:** If $K\lambda_{N-1} < 1$, $\|\boldsymbol{e}(t)\| \to 0$ as $t \to \infty$.
- **But:** Convergence may be not monotonic.

Global Exponential Stability (The "Directly")

- **Condition:** Monotonic need the **entire spectrum**.



Theorem (sTAC-2025): Global Exponential Stability Condition

Let $\alpha := K\max(|\lambda_1|, \lambda_{N-1})$. If $\alpha < 1$, the manifold $\mathcal{S}$ is **Globally Exponentially Stable**.

Proof: Using Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{e}^\top \boldsymbol{D}\boldsymbol{e}$ and Cauchy-Schwarz inequality.

Summing Up: The Recipe for Polarization

The Signal

- Polarization needs a **"Splitting Point"**: There must exist an unstable fixed point c° for the signal $s(x)$.

The Network

- The graph must be **not too much connected** (i.e., $\lambda_{N-1} > 0$).

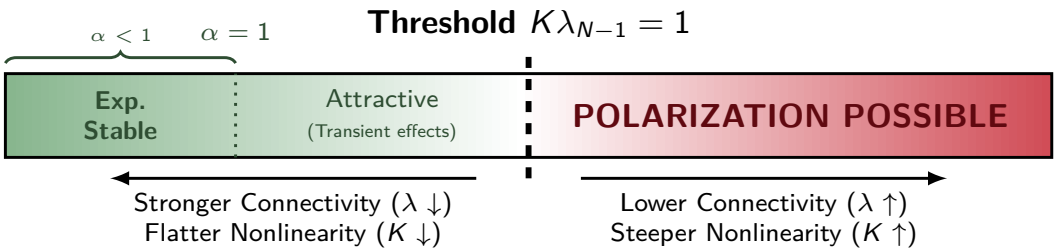
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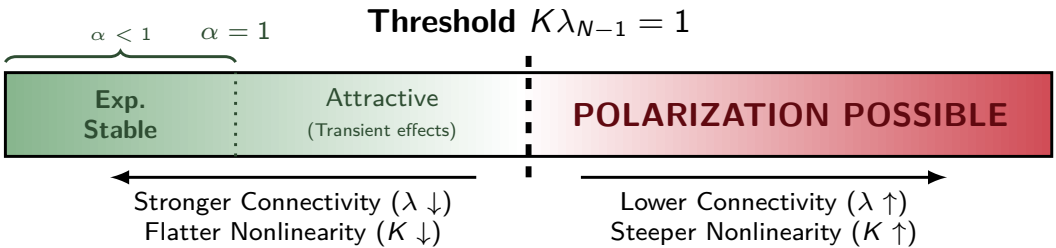
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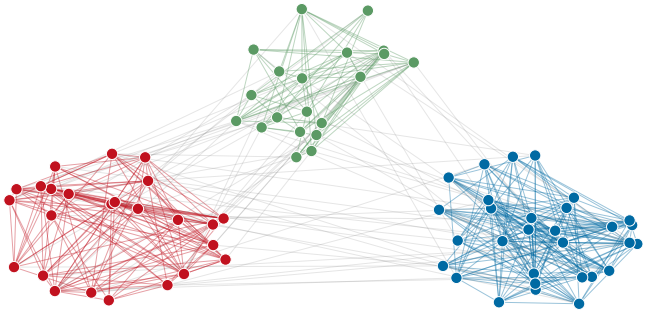
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We understand **the emergence of polarization** from the **global view**.

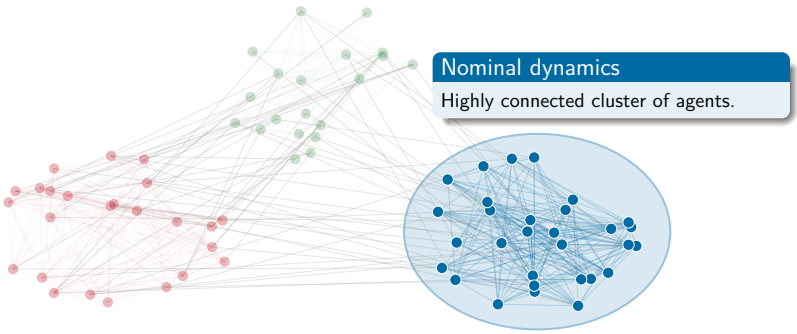
Understanding the Polarization in term of Graph

Where are the polarization clusters in the network?



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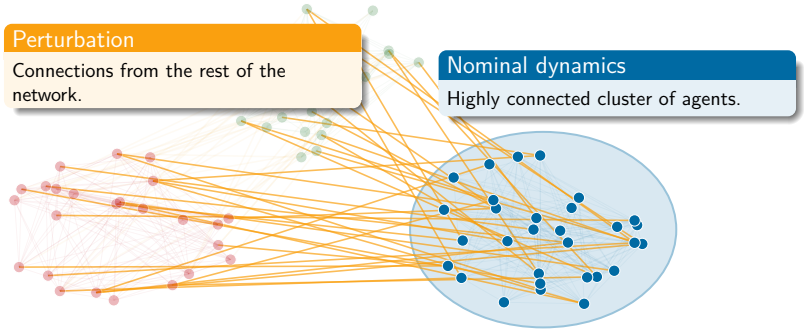


Definition (sTAC-2025): Internal Stability Condition

Cluster are maximal subsets $\mathcal{V}' \subset \mathcal{V}$ of connected agents such that the induced subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ verifies $\alpha_{\text{in}} = K \max(|\lambda_1^{\text{in}}|, \lambda_{N-1}^{\text{in}}) < 1$.

Understanding the Polarization in term of Graph

Where are the polarization clusters in the network?



Definition (sTAC-2025): Perturbation and Residual Perturbation

- **Perturbation** $\boldsymbol{p}(t)$: External influence from the rest network.
- **Residual** $\widetilde{\boldsymbol{p}}(t)$: Deviation from the average: $\widetilde{\boldsymbol{p}} = \boldsymbol{p} - \underbrace{\bar{p}\mathbf{1}}_{\text{Average}}$.

Robustness: Input-to-State Stability (ISS)

Theorem (sTAC-2025): ISS of the internal disagreement $\boldsymbol{e}(t)$

If $\alpha_{\text{in}} < 1$, the internal disagreement $\boldsymbol{e}(t)$ is bounded by the **heterogeneity** of external inputs $\widetilde{\boldsymbol{p}}$:

$$\limsup_{t\to\infty} \|\boldsymbol{e}(t)\| \leq \frac{1}{\alpha_{\text{in}}} \underbrace{\sup \|\widetilde{\boldsymbol{p}}(t)\|}_{\text{Non-uniformity}}$$

Proof: Using the Lypunov function $V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^\top \boldsymbol{D}\boldsymbol{e}$ with the previous result and the ISS definition (see (Khalil 2002)).

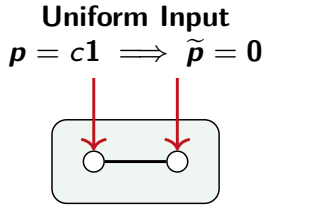
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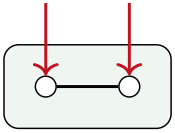
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Uniform Input

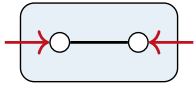
$\boldsymbol{p} = c\mathbf{1} \implies \widetilde{\boldsymbol{p}} = \mathbf{0}$



Cluster moves together ($\boldsymbol{e} \to 0$)

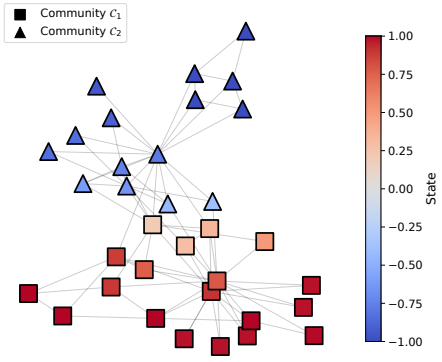
Heterogeneous Input

$\boldsymbol{p} \neq c\mathbf{1} \implies \widetilde{\boldsymbol{p}} \neq \mathbf{0}$



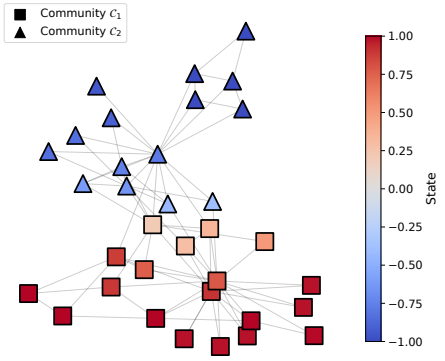
Internal Disagreement ($\boldsymbol{e} \not\to 0$)

Community detection and ISS properties

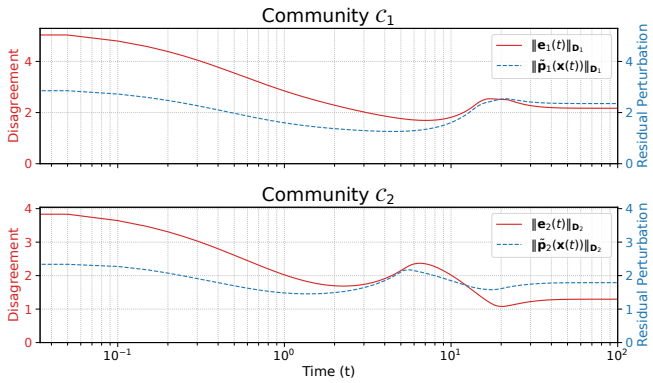


(a) Karate Club network with 2 communities.

Community detection and ISS properties

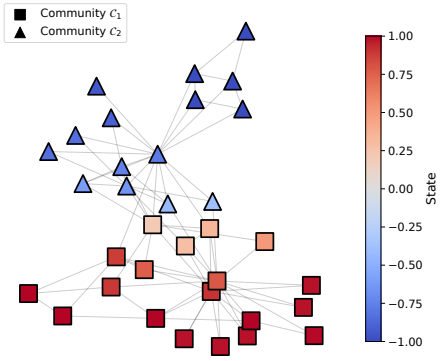


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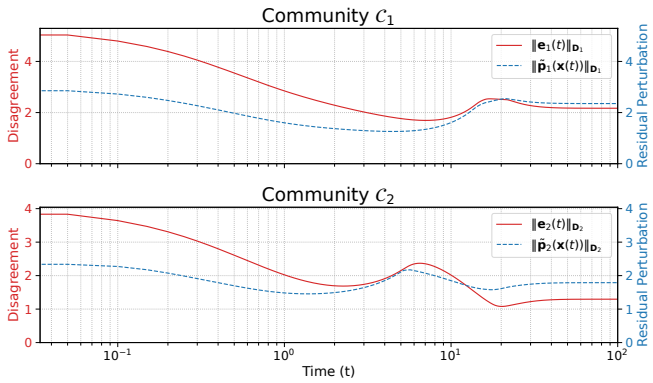


(b) ISS analysis of the 2 communities.

Community detection and ISS properties



(a) Karate Club network with 2 communities.



(b) ISS analysis of the 2 communities.

We understand **the polarization** using the **spatial structure** of the network.

Outline

1. Opinion Dynamics

2. Consensus Characterization and Analysis

3. Emergence of Polarization and Clusters

4. Coupling Opinion and Environment

5. Conclusion

Back to our model

- **Opinion Dynamics:** for agent i, the opinion dynamics is given by:

$$\dot{x}_i = -x_i + \beta r(E) + (1-\beta)\frac{1}{d_i}\sum_{j\in\mathcal{V}} a_{ij} s(x_j)$$

  - $\beta \in [0,1]$: Attention parameter.
  - $x_i \in [-1,1]$: Agent i opinion.

- **Environmental dynamics:** Physical process driving the evolution of the environment.

$$\tau_e \dot{E} = -\gamma E + u(\boldsymbol{x})$$

  - τ_e: time scale.
  - $u(\boldsymbol{x})$: Agents' collective impact.
  - γ: Restoring force of the environment.

Back to our model

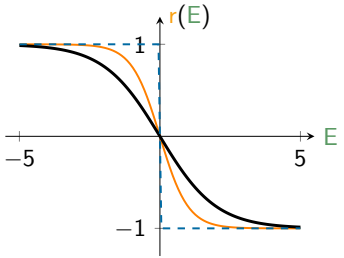
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  - If $E > 0$ (Pollution) $\to r < 0$ (Discourage).

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High state $E \to$ Negative feedback

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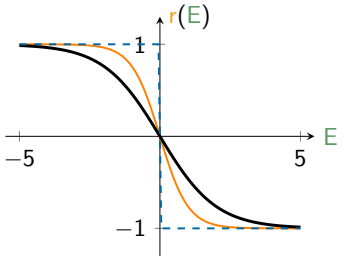
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  - Linear aggregation function $u : [-1,1]^N \to \mathbb{R}$.
  - **Degrades environment.**



High state $E \to$ Negative feedback

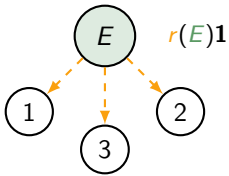
Model Reduction: The Power of Homogeneity

Proposition (CDC-2025): Forward Invariance of $\mathcal{S} \times \mathbb{R}$

Assume we start on the **Fully Synchronized Opinion (FSO)** manifold:

$$\mathcal{S} \times \mathbb{R} = \{(\boldsymbol{x}, E) \mid \boldsymbol{x} = p\mathbf{1}, E \in \mathbb{R}\}$$

Proof: The environment acts as a **uniform perturbation** ($\tilde{\boldsymbol{p}} = \mathbf{0}$).



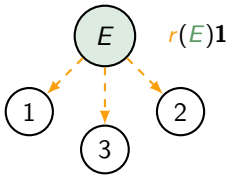
Since E pushes everyone equally, no disagreement is created.

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- **The common state is $p\mathbf{1}$.**

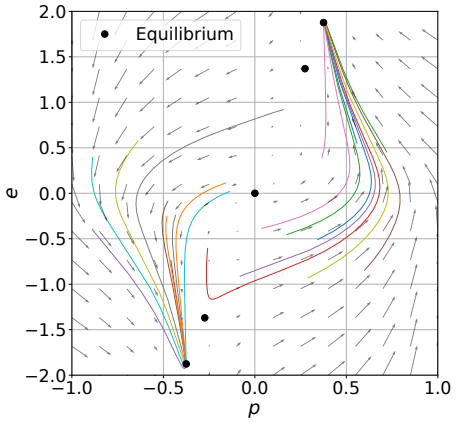
Dynamics on $\mathcal{S} \times \mathbb{R}$:

Since $\boldsymbol{x}(t)$ remains of the form $p(t)\mathbf{1}$, the N equations collapse to **one**:

$$\begin{cases} \dot{p} = -p + \beta r(E) + (1-\beta)s(p) \\ \tau_e \dot{E} = -\gamma E + u(p). \end{cases}$$

Numerical Observation of the Phenomenon

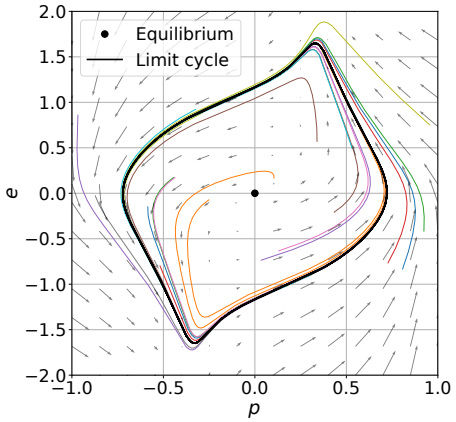
What happens when we vary the attention to the environment (β)?



1. Low β ($\beta = 0.24$): Convergence to Equilibria (Bistability).

Numerical Observation of the Phenomenon

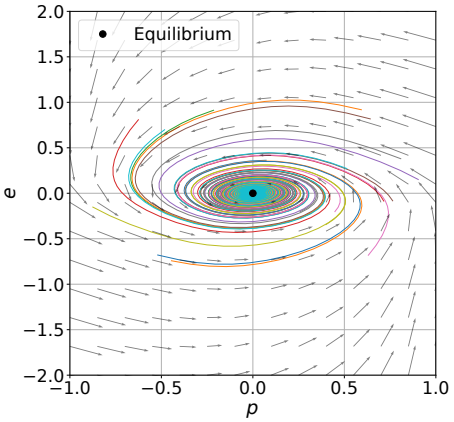
What happens when we vary the attention to the environment (β)?



2. Critical β ($\beta = 0.25$): Oscillations Emerge (Limit Cycle).

Numerical Observation of the Phenomenon

What happens when we vary the attention to the environment (β)?



3. High β ($\beta = 0.61$): Return to Equilibrium (Global Consensus).

Mathematical Mechanism: Pitchfork and Hopf Bifurcations

Question:

What happens when we vary the attention to the environment (β) for equilibrium $(0, 0)$?

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Theorem (CDC-2025): equilibria emergence

Pitchfork Bifurcation

- **Condition:** One eigenvalue $\lambda = 0$.
- **Cause:** Amplification $(1 - \beta)s'(0) > 1$.



Creates polarized Pro/Anti states.

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Theorem (CDC-2025): limit cycle emergence

Hopf Bifurcation

- **Condition:** Eigenvalues $\lambda = \pm i\omega$.
- **Cause:** Frustration Trace$(\boldsymbol{J}) = 0$.

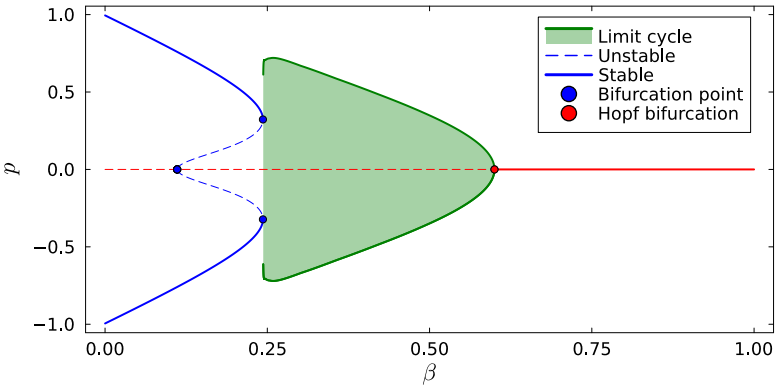


Creates predator-prey oscillations.

Proofs: Lyapunov-Schmidt reduction on point making the Jacobian singular. Transversality condition on the eigenvalues.

The Global Picture: Bifurcation Diagram

Summarizing all behaviors in a single **Bifurcation Diagram**.

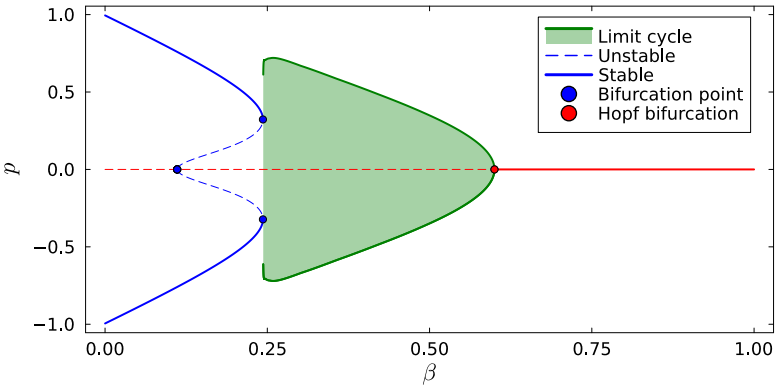


Regimes:

1. **Blue Lines:** Polarization.
2. **Green Area:** Oscillations.
3. **Red Line:** Neutral State.

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We understand **the emergence of oscillations** from the **bifurcation analysis**.

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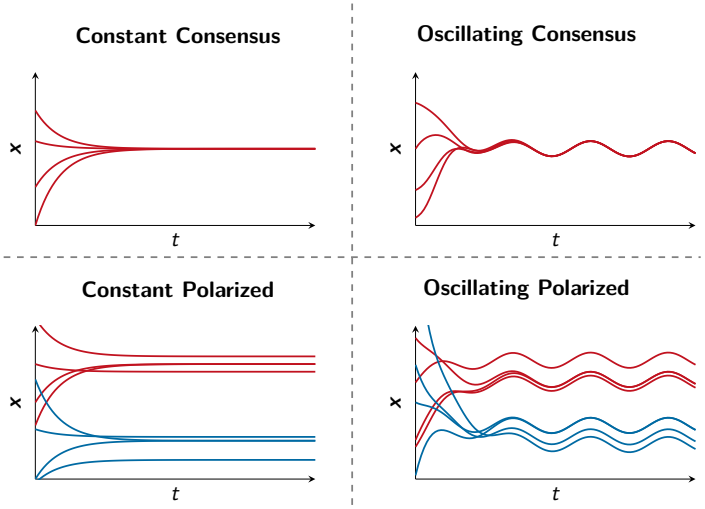
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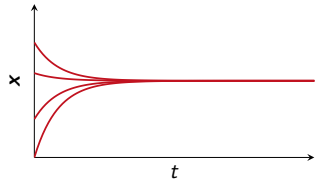


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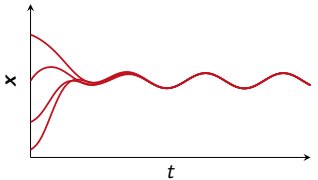
β close to 0 or 1

β in the Hopf interval

Constant Consensus



Oscillating Consensus



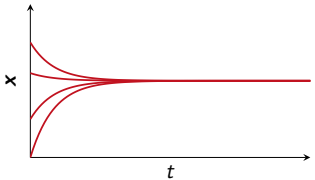
Constant Polarized

Oscillating Polarized

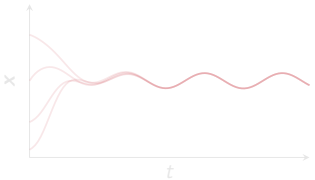
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Constant Consensus

$(K\lambda_{N-1} < 1)$
$\Rightarrow$ Synchronization

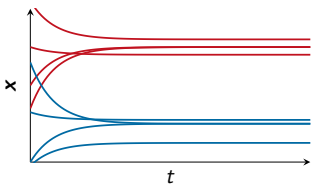


Oscillating Consensus



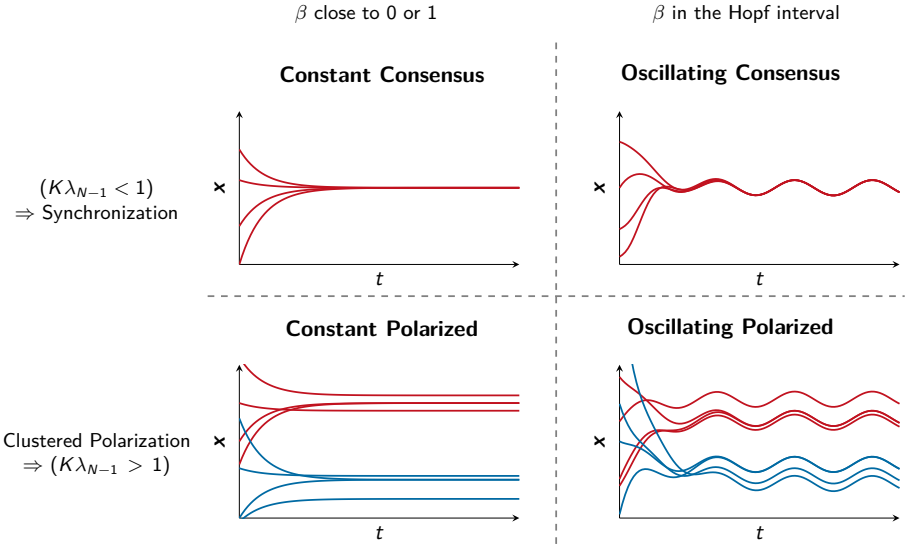
Constant Polarized

Clustered Polarization
$\Rightarrow (K\lambda_{N-1} > 1)$



Oscillating Polarized

Visual Conclusion: General Bifurcation Diagram



Conclusions: Summary of Contributions

1. A General Framework for Opinion Dynamics

Novelty: **Unified communication protocols** from DeGroot to complex OD and proved that strictly **positive graphs can polarize** via **non-local analysis**.

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Novelty: First formal characterization of **graph clusters as subgraphs that exponentially synchronize** under nonlinear interactions.

3. Networked Environmental Coupling

Novelty: Built the mathematical **knowledge to move beyond standard 2D mean-field simplifications** to the full network-environment feedback.

Perspectives: A Landscape of Open Questions

Climate & Games

- Endogenous preferences
- Economical modeling
- Stochastic climate events

Coupled Environments

- Multi-dimensional feedback
- Nonlinear bistable dynamics
- Coupled network-of-networks

Extended Framework

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- Dynamic / adaptive graphs
- Individual signal functions

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...but we can focus on two next steps:

1. Interconnected OD & Mode-Selective Bifurcations

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- Analyzing mode-selective bifurcations when environments couple to the network structure.

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2. Decentralized Community Detection via Social Learning

- Using the nonlinear OD framework as a **privacy-preserving, decentralized** clustering algorithm.
- Leveraging "structural splitting" so macroscopic communities emerge organically from local, peer-to-peer symmetry breaking.

Merci pour votre attention







- Altafini, Claudio (Apr. 2013). "Consensus Problems on Networks With Antagonistic Interactions". In: *IEEE Trans. Autom. Control* 58.4, pp. 935–946. ISSN: 1558-2523. DOI: 10.1109/TAC.2012.2224251.
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